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Enhancing Mechanisms of Neutrino Transitions in a Medium of Nonperiodic Constant - Density Layers and in the Earth

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Abstract

The enhancement of the transitions ν_μ (ν_e) $\rightarrow \nu_e$ ($\nu_{\mu;\tau}$), $\nu_2 \rightarrow \nu_e$, $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$, $\nu_e \rightarrow \nu_s$, etc. of neutrinos passing through a medium of nonperiodic density distribution, consisting of i) two layers of different constant densities (e.g., Earth core and mantle), or ii) three layers of constant density with the first and the third layers having identical densities and widths which differ from those of the second layer (e.g., mantle-core-mantle in the Earth), is studied. We find that, in addition to the known local maxima associated with the MSW effect and the neutrino oscillation length resonance (NOLR), the relevant two-neutrino transition probabilities have new resonance-like absolute maxima. The latter correspond to a new effect of *total* neutrino conversion and are absolute maxima in any independent variable: the neutrino energy, the layers widths, etc. We find the complete set of such maxima. We prove that the NOLR and the new absolute maxima are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. We show that the strong resonance-like enhancement of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos is due to the new effect of total neutrino conversion.

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1 Introduction

It was pointed out in [1] that the $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$ ($\nu_e \rightarrow \nu_{\mu(\tau)}$) oscillations of solar and atmospheric neutrinos in the Earth, caused by neutrino mixing (with nonzero mass neutrinos) in vacuum [2], can be strongly amplified by a new type of resonance-like mechanism which differs from the MSW one and takes place when the neutrinos traverse the Earth core on the way to the detector (see also [2, 3, 4]). As numerical calculations have shown, at small mixing angles ($\sin^2 2\theta \lesssim 0.05$), the maxima in the neutrino energy E of the corresponding transition probabilities, $P(\nu_2 \rightarrow \nu_e) \equiv P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$ ($P(\nu_e \rightarrow \nu_{\mu(\tau)})$), caused by the new enhancing mechanism, are absolute maxima and dominate in P_{e2} and $P(\nu_\mu \rightarrow \nu_e)$ ($P(\nu_e \rightarrow \nu_{\mu(\tau)})$): the values of the probabilities at these maxima in the simplest case of two-neutrino mixing are considerably larger - by a factor of $\sim (2.5 - 4.0)$ ($\sim (3.0 - 7.0)$), than the values of P_{e2} and $P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_{\mu(\tau)})$ at the local maxima associated with the MSW effect taking place in the Earth core (mantle). The effect of the new enhancement is equally dramatic at large mixing angles. The enhancement is present and dominates also in the $\nu_2 \rightarrow \nu_e$ transitions in the case of $\nu_e - \nu_s$ mixing and in the $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions at small mixing angles [1, 2, 3, 4]. Even at small mixing angles the enhancement is relatively wide [1, 2, 4] in the Nadir angle [3], h, and in the neutrino energy and leads to important observable effects. It also exhibits rather strong energy dependence.

The new resonance-like enhancement of the probability P_{e2} has important implications for the tests of the MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ and $\nu_e \rightarrow \nu_s$ transition solutions of the solar neutrino problem [1, 3, 5, 6, 7, 8, 9]. For values of Δm^2 from the small mixing angle (SMA) MSW solution region and the geographical latitudes at which the Super-Kamiokande, SNO and ICARUS detectors are located, the enhancement takes place in the $\nu_e \rightarrow \nu_{\mu(\tau)}$ case for values of the ^{8}B neutrino energy lying in the interval $\sim (5 - 12)$ MeV to which these detectors are sensitive. Accordingly, at small mixing angles the new resonance is predicted [5, 8] to produce a much bigger - by a factor of up to ~ 6 (~ 8), day-night, or D-N, asymmetry in the Super-Kamiokande (SNO) sample of solar neutrino (charged current) events, whose night fraction is due to the core-crossing neutrinos (*Core* D-N asymmetry), in comparison with the asymmetry determined by using the *whole night* event sample (*Night* asymmetry). As a consequence, it can be possible to test a substantial part of the MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ SMA solution region in the $\Delta m^2 - \sin^2 2\theta$ plane (see, e.g., [10]) by performing *core* D-N asymmetry measurements [5, 7, 8, 9]. The Super-Kamiokande collaboration has already successfully applied this approach to the analysis of their solar neutrino data [10]: the limit the collaboration has obtained on the D-N asymmetry utilizing only the measured event rate in the night N5 bin, approximately 80% of which is due to the Earth-core-crossing solar neutrinos, permitted to exclude a small part of the MSW SMA solution region. In contrast, the published Super-Kamiokande upper limit on

[2]The $\nu_2 \rightarrow \nu_e$ transition probability accounts, as is well-known, for the Earth effect in the solar neutrino survival probability in the case of the MSW two-neutrino $\nu_e \rightarrow \nu_{\mu(\tau)}$ and $\nu_e \rightarrow \nu_s$ transition solutions of the solar neutrino problem, ν_s being a sterile neutrino.

[3]The Nadir angle determines uniquely the neutrino trajectory through the Earth.

the *Night* D-N asymmetry [10] does not allow to probe the SMA solution region: the predicted asymmetry is too small (see, e.g., [5]).

The same new enhancement mechanism can and should be operative also [1] in the $\nu_\mu \to \nu_e$ ($\nu_e \to \nu_{\mu(\tau)}$) small mixing angle transitions of the atmospheric neutrinos crossing the Earth core if the atmospheric ν_μ and $\bar{\nu}_\mu$ indeed take part in large mixing vacuum $\nu_\mu(\bar{\nu}_\mu) \leftrightarrow \nu_\tau(\bar{\nu}_\tau)$, oscillations with $\Delta m^2 \sim (10^{-3} - 8 \times 10^{-3})$ eV^2, as is strongly suggested by the Super-Kamiokande data [11], and if all three flavour neutrinos are mixed in vacuum. The new resonance can take place practically for all neutrino trajectories through the core (e.g., , for the trajectories with $h = (0^\circ - 23^\circ)$). It can produce an excess of e-like events at $-1 \leq \cos\theta_z \leq -0.8$, θ_z being the Zenith angle, in the Super-Kamiokande multi-GeV atmospheric neutrino data and can be responsible for at least part of the strong Zenith angle dependence of the Super-Kamiokande multi-GeV and sub-GeV $\mu-$like data [1, 2, 4].

The Earth enhancement of the two-neutrino transitions of interest of the solar and atmospheric neutrinos at relatively small mixing angles has been discussed rather extensively, see, e.g., refs. [7, 12, 13, 14]. Some of the articles contain plots of the probabilities P_{e2} and/or $P(\nu_\mu \to \nu_e)$ or $P(\nu_e \to \nu_{\mu(\tau)})$ on which one can clearly recognize now the dominating maximum due to the new enhancement effect. However, this maximum was invariably interpreted to be due to the MSW effect in the Earth core (see, e.g., [12, 13]) before the appearance of [1].

In view of the important role the new mechanism of enhancement of the neutrino transitions in the Earth can play in the interpretation of the results of the experiments with solar and atmospheric neutrinos and in obtaining information about possible small mixing angle ν_μ ($\bar{\nu}_\mu$) and ν_e ($\bar{\nu}_e$) oscillations, it would be desirable to have an unambiguous understanding of the underlying physics of the mechanism. In [1] it was interpreted as an effect of constructive interference between the various probability amplitudes (notably of the neutrino transitions in the Earth mantle and in the Earth core), entering into the sum representing the probability amplitude of the transition in the Earth.

The exact conditions for the enhancement of the probabilities P_{e2}, $P(\nu_\mu \to \nu_e)$ ($P(\nu_e \to \nu_{\mu(\tau)})$), etc. by the new mechanism can be studied analytically [1] owing to the results of a detailed numerical analysis [15] (see also [14]) which showed that for the calculation of the probabilities of interest the two-layer model of the Earth density distribution provides a very good (in many cases - excellent) approximation to the more complicated density distributions predicted by the existing models of the Earth [16, 17]. In the two-layer model, the neutrinos crossing the Earth mantle and the core traverse effectively two layers with constant but different densities, $\bar{\rho}_{man}$ and $\bar{\rho}_c$, and chemical composition or electron fraction numbers, Y_e^{man} and Y_e^c. The neutrino transitions of interest [4] result from two-neutrino oscillations taking place i) first in the mantle over a distance X' with a mixing angle θ'_m and oscillation length L_{man}, ii) then in the core over a distance X'' with mixing angle $\theta''_m \neq \theta'_m$ and oscillation length $L_c \neq L_{man}$, and iii) again in the mantle over a distance X' with θ'_m and L_{man}. Utilizing the above prescription, analytic expressions for the probabilities of

[4] The densities $\bar{\rho}_{man,c}$ should be considered as mean densities along the neutrino trajectories.

neutrino transitions in the Earth P_{e2}, $P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_{\mu(\tau)})$, etc. were derived and were used to study their extrema [1]. Assuming that the neutrino oscillation parameters $\Delta m^2 > 0$ and $\cos 2\theta > 0$ are fixed and treating the neutrino paths X' and X'' as independent variables, it was found in [1] that in addition to the maxima corresponding to the MSW effect in the Earth mantle or in the Earth core, there exists a new maximum in P_{e2} and $P(\nu_\mu \to \nu_e)$. The latter takes place when the relative phases the states of the two energy-eigenstate neutrinos acquire after the neutrinos have crossed the mantle, $\Delta E' X' = 2\pi X'/L_{man}$, and the core, $\Delta E'' X'' = 2\pi X''/L_c$, obey the constraints,

$$\Delta E' X' = \pi(2k+1), \quad \Delta E'' X'' = \pi(2k'+1), \quad k, k' = 0, 1, 2, ... \,, \tag{1}$$

and if the inequality [1, 3]

$$\cos(2\theta''_m - 4\theta'_m + \theta) \; < 0, \tag{2}$$

is fulfilled. Condition (2) is valid for the probability P_{e2}. When equalities (1) hold, (2) ensures that P_{e2} has a maximum. At the maximum P_{e2} takes the form [1]

$$P^{max}_{e2} = \sin^2(2\theta''_m - 4\theta'_m + \theta). \tag{3}$$

The analogs of eqs. (1) - (3) for the probability $P(\nu_{\mu(e)} \to \nu_{e(\mu;\tau)})$ can be obtained [1] by formally setting $\theta = 0$ while keeping $\theta'_m \neq 0$ and $\theta''_m \neq 0$ in (1) - (3) [5]. The term "neutrino oscillation length resonance" (NOLR) was used in [1] to denote the resonance-type enhancement of P_{e2}, $P(\nu_\mu \to \nu_e)$ ($P(\nu_e \to \nu_{\mu(\tau)})$), etc. associated with the conditions (1) - (2).

The new mechanism of strong enhancement of the probabilities P_{e2}, $P(\nu_\mu \to \nu_e)$ ($P(\nu_e \to \nu_{\mu(\tau)})$), etc. for the Earth core crossing neutrinos, was identified in [1] with the neutrino oscillation length resonance. This interpretation was based on the observation that for the several fixed test values of $\sin^2 2\theta \lesssim 0.02$ and Nadir angle h considered in [1], conditions (1) were approximately satisfied at the corresponding non-MSW absolute maxima of P_{e2} and $P(\nu_\mu \to \nu_e)$ in the variable $E/\Delta m^2$ and the numerically calculated values of P_{e2} and $P(\nu_\mu \to \nu_e)$ at these maxima were reproduced by eq. (3) with a relatively good precision. Doubts about the correctness of such an interpretation remained since i) for the parameters of the Earth [16] (core radius, mantle width, etc.), used in the calculations, the set of conditions (1) - (2), as was shown in [1], cannot be exactly satisfied at small mixing angles, and ii) although the phase $\Delta E' X'$ was close to π at the relevant maxima of P_{e2} and $P(\nu_\mu \to \nu_e)$ in the test cases studied, the values of the phase $\Delta E'' X''$ differed quite substantially from those required by eq. (1). Moreover, the same (or similar) enhancement mechanism was found to be operative in the $\nu_2 \to \nu_e$ transitions in the case of $\nu_e - \nu_s$ mixing and in the $\nu_e \to \nu_s$ transitions, at small mixing angles [1]. However, as was noticed in [1], the conditions (1) - (2) are not even approximately fulfilled for the $\nu_2 \to \nu_e \cong \nu_s \to \nu_e$

[5] The conditions for the corresponding maximum in $P(\nu_e \to \nu_s)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_s)$ coincide in form with those for $P(\nu_{\mu(e)} \to \nu_{e(\mu;\tau)})$.

and $\nu_e \to \nu_s$ transitions at small mixing angles; similar conclusions were reached for the $\bar{\nu}_\mu \to \bar{\nu}_s$ transitions [2].

The indicated results stimulated the systematic study of the extrema of the probabilities P_{e2}, $P(\nu_\mu \to \nu_e)$, $P(\nu_e \to \nu_s)$, etc. of the neutrino transitions in the Earth and more generally, in a medium of non-periodic constant density layers, which is presented here. We consider transitions of neutrinos in a medium consisting of i) two layers of different constant densities, and ii) three layers of constant density with the first and the third layers having identical densities and widths which differ from those of the second layer. Neutrinos pass through such systems of layers on the way to the neutrino detectors, e.g., when they i) are produced in the central region of the Earth and traverse both the Earth core and mantle, ii) travel first in vacuum and then in the Earth mantle, iii) cross the Earth mantle, the core and the mantle again. We derive and analyze the complete set of extrema of the two-neutrino transition probabilities in both cases of a medium with two and three constant density layers, assuming that [1] $\Delta m^2/E$ and $\sin^2 2\theta$ are fixed and treating the widths of the layers, X' and X'', as independent variables. For both media considered we find that in addition to the local maxima associated with the MSW effect and the NOLR, there exist absolute maxima caused by a new effect of enhancement and corresponding to a total neutrino conversion. We find the complete set of such absolute maxima. The latter are absolute maxima in any possible independent variables: the neutrino energy, the widths of one of the layers, etc. The conditions for existence of the new maxima are derived and it is shown that they are fulfilled, in particular, for the transitions of the Earth-core-crossing solar and atmospheric neutrinos. These conditions differ from the conditions of enhancement of the probability of transitions of neutrinos propagating in a medium with periodically varying density, discussed in [18]. We show that the strong resonance-like enhancement of these transitions is due to the effect of total neutrino conversion. At small mixing angles and in the case of the transitions $\nu_2 \to \nu_e$ and $\nu_\mu \to \nu_e$, for instance, the values of the parameters $\sin^2 2\theta$ and $\Delta m^2/E$ at which the total neutrino conversion takes place for neutrinos traversing the Earth core along a given trajectory are rather close to the values of the parameters for which the NOLR giving $P_{e2} = 1$ or $P(\nu_\mu \to \nu_e) = 1$ occurs, and the two enhancement mechanisms practically coincide. For smaller $\sin^2 2\theta$ the NOLR reproduces approximately (in some cases - roughly) the enhancement: for fixed h and $\sin^2 2\theta = (10^{-3} - 10^{-2})$, for instance, it reproduces the values of $P(\nu_2 \to \nu_e)$ and $P(\nu_\mu \to \nu_e)$ at the absolute maxima in the $\Delta m^2/E$ variable with an error of $\sim$ (15 - 60)% (see further). In the case of the $\nu_2 \to \nu_e \cong \nu_s \to \nu_e$, $\nu_e \to \nu_s$ and the $\bar{\nu}_\mu \to \bar{\nu}_s$ transitions at small mixing angles, the total neutrino conversion occurs for values of the parameters for which the NOLR is not realized. In all transitions, however, only the total neutrino conversion mechanism is operative. This mechanism is responsible also for the strong enhancement of the above transitions at large mixing angles as well as of the $\nu_2 \to \nu_e$ transitions due to $\nu_e - \nu_s$ mixing and of the $\nu_e \to \nu_s$ and the $\bar{\nu}_\mu \to \bar{\nu}_s$ transitions at small and large mixing angles.

We demonstrate that the NOLR and the newly found resonance-like enhancement effect are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. Thus, the local and abso-

lute maxima they produce in the neutrino transition probabilities are of interference nature. Therefore we confirm the physical interpretation of the strong resonance-like enhancement of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos, given in [1].

Most of the results of our analysis are illustrated by the physically realistic examples of the transitions of neutrinos passing through the Earth. These examples and the results regarding the transitions of Earth-core-crossing solar and atmospheric neutrinos are derived using the Stacey model from 1977 [16] as a reference Earth model. The density distribution in the Stacey model is spherically symmetric and in addition to the two major density structures - the core and the mantle, there are a certain number of substructures (shells or layers). The core has a radius $R_c = 3485.7$ km, the Earth mantle depth is approximately $R_{man} = 2885.3$ km, and the Earth radius in the Stacey model is $R_\oplus = 6371$ km. Therefore, when the Nadir angle h is less than $33.17°$, neutrinos arriving at the detector pass through three layers: the mantle, the core and the mantle again. The mean values of the matter densities of the core and of the mantle read, respectively: $\bar{\rho}_c \cong 11.5$ g/cm^3 and $\bar{\rho}_{man} \cong 4.5$ g/cm^3. The density distribution in the 1977 Stacey model practically coincides with that in the more recent PREM model [17]. For Y_e we have used the standard values [16, 17, 19] (see also [5]) $Y_e^{man} = 0.49$ and $Y_e^c = 0.467$.

A brief description of the results of the present study is given in ref. [20].

2 Preliminary Remarks

We will consider the simple case of mixing of two weak-eigenstate neutrinos ν_α and ν_β, $\alpha \neq \beta = e, \mu, \tau, s$, ν_s being a sterile neutrino, in vacuum. The mixing matrix relating the states of the neutrinos $\nu_{\alpha,\beta}$ and of the neutrinos $\nu_{1,2}$ having definite masses $m_{1,2}$ in vacuum is chosen in the form:

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \tag{4}$$

where θ is the vacuum mixing angle. In the case of relativistic and stable neutrinos $\nu_{1,2}$, the evolution equation for neutrino propagation in constant-density medium reads

$$i\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \frac{\Delta E}{2} \begin{pmatrix} -\cos(2\theta_m) & \sin(2\theta_m) \\ \sin(2\theta_m) & \cos(2\theta_m) \end{pmatrix} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \mathcal{M} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}. \tag{5}$$

Here θ_m is the mixing angle in matter,

$$\cos(2\theta_m) = \frac{1}{\Delta E}\ ((\Delta m^2/2E)\cos(2\theta) - V_{\alpha\beta}), \tag{6}$$

E being the neutrino energy, ΔE is the difference between the energies of the two energy-eigenstate neutrinos in matter,

$$\Delta E = \frac{\Delta m^2}{2E}\sqrt{\left(\cos(2\theta) - \frac{2EV_{\alpha\beta}}{\Delta m^2}\right)^2 + \sin^2(2\theta)}, \tag{7}$$

$\Delta m^2 = m_2^2 - m_1^2$, and $V_{\alpha\beta}$ is the difference between the effective potentials of ν_α and ν_β in the medium. We shall always assume in what follows that

$$\cos(2\theta) > 0, \quad \Delta m^2 > 0. \tag{8}$$

In the case of neutrino propagation in an electrically neutral unpolarized cold medium, like the Earth, one has:

$$V_{e\mu} = \sqrt{2}G_F N_e, \quad V_{es} = \sqrt{2}G_F(N_e - \frac{1}{2}N_n), \quad V_{\mu s} = -\sqrt{2}G_F N_n/2, \tag{9}$$

where N_e and N_n are the electron and the neutron number densities in the medium, respectively. The antineutrino states $\bar{\nu}_\alpha$ and $\bar{\nu}_\beta$ satisfy the same equations (5) with $V_{\alpha\beta}$ replaced by $V_{\bar{\alpha}\bar{\beta}} = -V_{\alpha\beta}$ in eqs. (6) and (7). For the Earth $N_e \cong N_n$ and in addition to $V_{e\mu} > 0$ we have $V_{es} > 0$ and $V_{\bar{\mu}\bar{s}} > 0$. If (8) holds, the neutrino mixing can be enhanced by the Earth matter only in the cases of ν_μ (ν_e) $\to \nu_e$ ($\nu_{\mu;\tau}$) ($\nu_2 \to \nu_e$), $\nu_e \to \nu_s$ and $\bar{\nu}_\mu \to \bar{\nu}_s$ transitions, while for, e.g., $\cos(2\theta) > 0$ and $\Delta m^2 < 0$ this will be true for the $\bar{\nu}_\mu$ ($\bar{\nu}_e$) $\to \bar{\nu}_e$ ($\bar{\nu}_{\mu;\tau}$) ($\bar{\nu}_2 \to \bar{\nu}_e$), $\bar{\nu}_e \to \bar{\nu}_s$ and $\nu_\mu \to \nu_s$ transitions. Our general results will be formulated for the generic weak-eigenstate neutrino transitions, $\nu_\alpha \to \nu_\beta$; for neutrinos crossing the Earth we will be interested either in the former or in the latter set of transitions, depending on whether eq. (8) holds or $\Delta m^2 \cos(2\theta) < 0$. The case of (solar) $\nu_2 \to \nu_e$ transitions in a three-layer medium (the Earth) will be considered separately.

The evolution of the neutrino system is given by the unitary matrix (see, e.g., [21, 22])

$$U = \exp(-i\mathcal{M}t) = \cos\phi - i(\boldsymbol{\sigma}\boldsymbol{n})\sin\phi, \tag{10}$$

where

$$\phi = \frac{1}{2}\Delta E\ t \tag{11}$$

and

$$\mathbf{n} = (\sin(2\theta_m), 0, -\cos(2\theta_m)) \tag{12}$$

is a real unit vector. The probabilities of the $\nu_\alpha \to \nu_\beta$ ($\nu_\beta \to \nu_\alpha$) transition, $P_{\alpha\beta(\beta\alpha)} = |A_{\alpha\beta(\beta\alpha)}|^2$, and of the ν_α (ν_β) survival, $P_{\alpha\alpha(\beta\beta)} = |A_{\alpha\alpha(\beta\beta)}|^2$, are determined by the elements of the evolution matrix U, which coincide with the four different probability amplitudes $A_{\alpha\beta} = U_{\beta\alpha}$, etc.:

$$P_{\alpha\beta} = |U_{\beta\alpha}|^2 = (n_1\sin\phi)^2 + (n_2\sin\phi)^2 = P_{\beta\alpha} \tag{13}$$

and

$$P_{\alpha\alpha} = |U_{\alpha\alpha}|^2 = \cos^2\phi + (n_3\sin\phi)^2 = 1 - P_{\alpha\beta} = P_{\beta\beta}. \tag{14}$$

We are interested in the extrema of the transition probability $P_{\alpha\beta}$ when neutrinos propagate through a medium with (nonperiodic sequence of) layers of different constant density and chemical composition. In this case the evolution matrix, as is well-known, represents a product of the evolution matrices for the different layers and can always be written in the same form as in eq. (10). It follows from (13) and (14)

that the conditions for an *absolute* maximum of the transition probability, $P_{\alpha\beta} = 1$, read

$$max\ P_{\alpha\beta} = 1: \ \begin{cases} \cos\phi = 0 \\ n_3 \sin\phi = 0, \end{cases} \tag{15}$$

while the conditions for an *absolute* minimum of $P_{\alpha\beta}$ have the form:

$$min\ P_{\alpha\beta} = 0: \ \begin{cases} n_1 \sin\phi = 0 \\ n_2 \sin\phi = 0. \end{cases} \tag{16}$$

When neutrinos propagate in a constant-density medium the transition probability

$$P_{\alpha\beta} = \frac{1}{2}(1 - \cos 2\phi)\sin^2(2\theta_m) = (\sin\phi \sin(2\theta_m))^2 \tag{17}$$

can be non-negligible even in the case of small vacuum mixing angle θ if the neutrino mixing in matter is maximal, $\theta_m = \pi/4$. This is the well-known MSW effect which takes place when the resonance condition

$$\cos(2\theta_m) = 0 \tag{18}$$

is fulfilled. In order to get total neutrino conversion, $P_{\alpha\beta} = 1$, the additional requirement

$$\cos\phi = 0 \tag{19}$$

has to be satisfied. For a fixed time of propagation t, or a distance $X \cong t$ traveled by the neutrino, there always exists a solution of the system of equations (15) The *absolute* minima $P_{\alpha\beta} = 0$ correspond, as it follows from (16), to the curves

$$\sin\phi = 0. \tag{20}$$

In the next sections we discuss the case of neutrino propagation through a number of nonperiodic constant-density layers. A new phenomenon of maximal constructive interference between the probability amplitudes of the transitions in the different layers takes place in this case, leading to a substantial enhancement of the transition probability. Even when the oscillation parameters have values very different from those required by the resonance conditions (18) for the individual layers, the neutrino transition probability can reach its absolute maximum, $P_{\alpha\beta} = 1$, due to this effect.

3 Medium with Two Constant - Density Layers

Suppose neutrinos ν_α or ν_β have crossed a medium consisting of two layers having constant but different density and chemical composition. These could be the Earth core and mantle (for neutrinos born in the Earth central region), the vacuum and the Earth mantle, etc. Let us denote the effective potential differences and the lengths of the paths of the neutrinos (antineutrinos) in the two layers by $V'_{\alpha\beta}$ ($V'_{\bar{\alpha}\bar{\beta}}$), $V''_{\alpha\beta}$ ($V''_{\bar{\alpha}\bar{\beta}}$) and X', X'', respectively. We shall assume without loss of generality that

$$0 \leq |V'_{\alpha\beta(\bar{\alpha}\bar{\beta})}| < |V''_{\alpha\beta(\bar{\alpha}\bar{\beta})}|. \tag{21}$$

The transitions of ν_α and ν_β crossing the two layers will be determined by the two sets of two parameters - the mixing angle in matter (6) or the unit vector (12), and the phase difference (11) with $t = X'$ or X'', which characterize the transitions in each of the layers: θ'_m or $\mathbf{n}'$ and ϕ', and θ''_m or $\mathbf{n}''$ and ϕ''. Since the evolution matrix of the system is given by $U = U''U'$, U' and U'' being the evolution matrices in the first and in the second layers, the parameters of U, Φ and $\mathbf{n}$ (eq. (10)), can be expressed in terms of ϕ', $\mathbf{n}'$ and ϕ'', $\mathbf{n}''$ as follows (see, e.g., [21]):

$$\begin{cases} \cos\Phi = \cos\phi'\cos\phi'' - (\mathbf{n}'\cdot\mathbf{n}'')\sin\phi'\sin\phi'' \\ = \cos\phi'\cos\phi'' - \cos(2\theta''_m - 2\theta'_m)\sin\phi'\sin\phi'', \\ \mathbf{n}\sin\Phi = \mathbf{n}'\sin\phi'\cos\phi'' + \mathbf{n}''\cos\phi'\sin\phi'' - [\mathbf{n}'\times\mathbf{n}'']\sin\phi'\sin\phi''. \end{cases} \tag{22}$$

Using, e.g., eqs. (13) and (22) it is not difficult to derive the transition probability $P_{\alpha\beta} = P_{\beta\alpha}$ in this case:

$$P_{\alpha\beta} = \frac{1}{2}\left[1-\cos 2\phi'\right]\sin^2 2\theta'_m + \frac{1}{2}\left[1-\cos 2\phi''\right]\sin^2 2\theta''_m + \frac{1}{4}\left[1-\cos 2\phi'\right]\left[1-\cos 2\phi''\right]\times$$

$$\times\left[\sin^2(2\theta''_m - 2\theta'_m) - \sin^2 2\theta'_m - \sin^2 2\theta''_m\right] + \frac{1}{2}\sin 2\phi'\ \sin 2\phi''\ \sin 2\theta'_m\sin 2\theta''_m. \tag{23}$$

Due to T-invariance and unitarity, the probability (23) is symmetric with respect to the interchange of the parameters of the two layers, i.e., does not depend on the order in which the neutrinos traverse them.

For fixed $\sin^2(2\theta)$ and $\Delta m^2/E$ and given density and chemical composition in the two layers, the unit vectors $\mathbf{n}'$ and $\mathbf{n}''$ are also fixed for each layer. The phases ϕ' and ϕ'', which depend on the widths of the layers X' and X'', can be independent variables of the system if the two widths X' and X'' can be varied independently. This is an interesting case with a clear physical meaning and we are going to investigate it in the present and the next Sections. In this way the NOLR resonance in the $\nu_2 \to \nu_e$, ν_μ (ν_e) $\to \nu_e$ ($\nu_{\mu;\tau}$) and $\nu_e \to \nu_s$ transitions was discovered [1]. Let us note that in the case of the Earth, X' and X'' cannot be treated as independent variables as long as the Earth radius and the Earth core radius are fixed. Neutrino transitions in the Earth will be considered at the end of this Section and in Section 6.

Varying the phases ϕ' and ϕ'' we can investigate the number and the structure of the extrema of the transition probability $P_{\alpha\beta}$. These are determined by the system of two equations

$$\frac{dP_{\alpha\beta}}{d\phi'} = \sin 2\theta'_m\ F(2\theta'_m - 2\theta''_m, 2\theta''_m; 2\phi', 2\phi'') = 0, \tag{24}$$

$$\frac{dP_{\alpha\beta}}{d\phi''} = \sin 2\theta''_m\ F(2\theta''_m - 2\theta'_m, 2\theta'_m; 2\phi'', 2\phi') = 0, \tag{25}$$

where

$$F(Y, Z; \varphi, \psi) = \sin Y\cos Z\sin\varphi + \sin Z\left[\sin\varphi\cos\psi\cos Y + \cos\varphi\sin\psi\right], \tag{26}$$

and by two known supplementary conditions on the values of the second derivatives of $P_{\alpha\beta}$ at the points where eqs. (24) and (25) hold.

We are interested first of all in the *absolute* maxima of the neutrino conversion,

$$case\ A: \quad max\ P_{\alpha\beta} = 1. \tag{27}$$

They are determined by the equations (15) and (22)

$$max\ P_{\alpha\beta} = 1: \quad \left\{ \begin{array}{l} \cos\Phi \equiv \cos\phi' \cos\phi'' - \cos(2\theta''_m - 2\theta'_m)\sin\phi' \sin\phi'' = 0 \\ n_3 \sin\Phi \equiv -[\cos(2\theta'_m)\sin\phi'\cos\phi'' + \cos(2\theta''_m)\cos\phi'\sin\phi''] = 0. \end{array} \right. \tag{28}$$

It is not difficult to check that conditions (24) and (25) are satisfied if relations (28) hold. The solutions of (28) can be readily found:

$$solution\ A: \quad \left\{ \begin{array}{l} \tan\phi' = \pm\sqrt{\dfrac{-\cos(2\theta''_m)}{\cos(2\theta'_m)\cos(2\theta''_m - 2\theta'_m)}}\ , \\ \tan\phi'' = \pm\sqrt{\dfrac{-\cos(2\theta'_m)}{\cos(2\theta''_m)\cos(2\theta''_m - 2\theta'_m)}}\ , \end{array} \right. \tag{29}$$

where the signs are correlated. Obviously, the solutions (29) do not exist for any pairs of values of the parameters $\sin^2(2\theta)$ and $\Delta m^2/E$. Under the assumptions (8), (21) and for $V'_{\alpha\beta} \geq 0$ (e.g., for the ν_μ (ν_e) $\rightarrow \nu_e$ ($\nu_{\mu;\tau}$), $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions of the Earth-crossing neutrinos), they can take place only in the region limited by the three conditions [6] (Fig. 1)

$$region\ A: \quad \left\{ \begin{array}{l} \cos 2\theta'_m \geq 0 \\ \cos 2\theta''_m \leq 0 \\ \cos(2\theta''_m - 2\theta'_m) \geq 0. \end{array} \right. \tag{30}$$

The first boundary, $\cos 2\theta'_m = 0$, corresponds to a *total* neutrino conversion *in the first layer*. From (29) we get the correct result for the relevant conditions on the phases (see (19) and (20)),

$$solution\ B: \quad \left\{ \begin{array}{l} \cos\phi' = 0,\ \ or\ 2\phi' = \pi(2k'+1),\ \ k' = 0,1,..., \\ \sin\phi'' = 0,\ \ or\ 2\phi'' = 2\pi k'',\ \ k'' = 0,1,..., \end{array} \right. \tag{31}$$

guarantying that i) neutrino conversion takes place in the first layer, and ii) no neutrino conversion occurs in the second layer, so that when $\cos 2\theta'_m = 0$ we have maximal transition probability, $P_{\alpha\beta} = 1$. It is easy to show by direct calculations using eqs. (23) - (25) that the phase conditions (31) correspond to the local maxima

$$case\ B: \ \ max\ P_{\alpha\beta} = \sin^2 2\theta'_m, \tag{32}$$

provided the oscillation parameters $\sin^2(2\theta)$ and $\Delta m^2/E$ belong to the region B,

$$region\ B: \quad \cos 2\theta'_m \leq 0. \tag{33}$$

[6]These conditions can also be derived from the two supplementary inequalities ensuring that the solutions (29) of (24) and (25) indeed correspond to maxima of $P_{\alpha\beta}$.

It proves useful to interpret this result in terms of probability amplitudes. If neutrinos ν_α have crossed the two layers, the amplitude of the $\nu_\alpha \to \nu_\beta$ transition represents a sum of two terms:

$$A_{\alpha\beta} = U_{\beta\alpha} = U''_{\beta\alpha} U'_{\alpha\alpha} + U''_{\beta\beta} U'_{\beta\alpha}, \tag{34}$$

where

$$\begin{array}{ll} U'_{\alpha\alpha} = \cos\phi' + i\cos(2\theta'_m)\sin\phi', & U''_{\beta\alpha} = -i\sin(2\theta''_m)\sin\phi'', \\ U'_{\beta\alpha} = -i\sin(2\theta'_m)\sin\phi', & U''_{\beta\beta} = \cos\phi'' - i\cos(2\theta''_m)\sin\phi''. \end{array} \tag{35}$$

In the general case the transition probability

$$P_{\alpha\beta} = |U''_{\beta\alpha}|^2 |U'_{\alpha\alpha}|^2 + |U''_{\beta\beta}|^2 |U'_{\beta\alpha}|^2 + 2\mathrm{Re}\left(\left(U''_{\beta\alpha} U'_{\alpha\alpha}\right)^* U''_{\beta\beta} U'_{\beta\alpha}\right) \tag{36}$$

is a sum of two products of the probabilities of neutrino oscillations in the different layers and of the interference term. The latter can play a crucial role in the resonance-like enhancement of the neutrino transitions.

The phase requirements (31) reduce the expression in eq. (36) to only one term, $P_{\alpha\beta} = |U'_{\beta\alpha}|^2$, with no contribution from the interference term because $U''_{\beta\alpha} = 0$. Thus, the maxima in region B (33) can be ascribed to neutrino transitions taking place in the first layer only. The absolute maxima, max $P_{\alpha\beta} = 1$, correspond for $\sin^2 2\theta < 1$ to the MSW effect in the first layer.

The second boundary $\cos(2\theta''_m) = 0$ in (30) corresponds to a *total* neutrino conversion in the second layer,

$$solution\ C: \quad \left\{ \begin{array}{l} \sin\phi' = 0,\ or\ 2\phi' = 2\pi k',\ k' = 0, 1, ..., \\ \cos\phi'' = 0,\ or\ 2\phi'' = \pi(2k'' + 1),\ k'' = 0, 1, ...\ . \end{array} \right. \tag{37}$$

This case is completely analogous to the previously considered case B, with the two layers interchanged. Using eqs. (23) - (25) we get solution C which realizes the *local* maxima

$$case\ C: \quad max\ P_{\alpha\beta} = \sin^2 2\theta''_m, \tag{38}$$

in the region

$$region\ C: \quad \cos 2\theta''_m \geq 0. \tag{39}$$

The case of equality in eq. (39) can be associated with the MSW effect taking place in the second layer.

We get a very different mechanism of enhancement of the neutrino transition probability $P_{\alpha\beta}$ in the intermediate region

$$\left\{ \begin{array}{l} \cos 2\theta'_m > 0, \\ \cos 2\theta''_m < 0, \end{array} \right. \tag{40}$$

located between the regions B (33) and C (39). The maxima of $P_{\alpha\beta}$ in this region are caused by a maximal contribution of the interference term in eq. (36) for $P_{\alpha\beta}$. Indeed, using eq. (34) one can write the probability $P_{\alpha\beta}$ in the form:

$$P_{\alpha\beta} = |\mathbf{z}_1 + \mathbf{z}_2|^2, \tag{41}$$

where $\mathbf{z}_1 = (\mathrm{Re}(U''_{\beta\alpha}U'_{\alpha\alpha}), \mathrm{Im}(U''_{\beta\alpha}U'_{\alpha\alpha}))$ and $\mathbf{z}_2 = (\mathrm{Re}(U''_{\beta\beta}U'_{\beta\alpha}), \mathrm{Im}(U''_{\beta\beta}U'_{\beta\alpha}))$ are two vectors in the complex plane. In the case of the probability $P_{\alpha\beta}$ the interference is maximal and constructive, as it follows from (41), when $\mathbf{z}_1$ and $\mathbf{z}_2$ are collinear and point in the same direction, i.e., when

$$\frac{\mathrm{Im}(U''_{\beta\alpha}U'_{\alpha\alpha})}{\mathrm{Im}(U''_{\beta\beta}U'_{\beta\alpha})} = \frac{\mathrm{Re}(U''_{\beta\alpha}U'_{\alpha\alpha})}{\mathrm{Re}(U''_{\beta\beta}U'_{\beta\alpha})} > 0. \tag{42}$$

The second equation in (15) (or (28)) ensures that the two vectors are collinear, while the constraints (40) guarantee that they point in the same direction. Actually, the equation in (42) coincides with the second equation in (15) for the resonance condition (18)

$$\cos(2\theta_m) = \frac{1}{\sin\phi}\left[\cos(2\theta')\sin\phi'\cos\phi'' + \cos(2\theta'')\cos\phi'\sin\phi''\right] = 0, \tag{43}$$

with the additional constraints (40), corresponding to the intermediate region where the new type of enhancement takes place. The transition probability of interest (23) can obviously be represented in the two layer case under discussion also in the form (17),

$$P_{\alpha\beta} = \sin^2(2\theta_m)\sin^2\phi, \tag{44}$$

where the first and the second multipliers are defined by eq. (43) and the first equation in (22). To get a total neutrino conversion, $P_{\alpha\beta} = 1$, not only eq. (43) has to hold, but also condition (19) must be satisfied. This is possible in region A (30), but not in the whole intermediate region (40).

Indeed, the boundary $\cos(2\theta''_m - 2\theta'_m) = 0$ in (30) corresponds to a maximum of $P_{\alpha\beta}$ associated with the phase constraints

$$\textit{solution } D: \quad \begin{cases} \cos\phi' = 0,\ \textit{or}\ 2\phi' = \pi(2k'+1),\ k' = 0, 1, ..., \\ \cos\phi'' = 0,\ \textit{or}\ 2\phi'' = \pi(2k''+1),\ k'' = 0, 1, ...\ . \end{cases} \tag{45}$$

The above conditions are necessary for a maximal neutrino conversion in each layer (see (19) and (35)). As can be shown exploiting eqs. (23) - (25), they lead to local maxima

$$\textit{case } D: \quad \textit{max } P_{\alpha\beta} = \sin^2(2\theta''_m - 2\theta'_m), \tag{46}$$

if the oscillation parameters belong to the finite region D,

$$\textit{region } D: \quad \cos(2\theta''_m - 2\theta'_m) \leq 0. \tag{47}$$

Our results show a very interesting feature of the neutrino transitions in a medium consisting of two constant-density layers. One can have total neutrino conversion, $P_{\alpha\beta} = 1$, even if the MSW resonance does not take place in any of the two layers. As we have seen, this is possible in region A, excluding the two boundaries $\cos(2\theta'_m) = 0$ and $\cos(2\theta''_m) = 0$, corresponding to the MSW effect. Therefore, in order to have a large transition probability, $P_{\alpha\beta} \cong 1$, a periodic density profile is not required even when the MSW resonance is not realized in any of the two layers. These conclusions

remain valid, as we shall see, for transitions of neutrinos traversing three layers of constant density, e.g., for the transitions of the solar and atmospheric neutrinos crossing the Earth core on the way to the detectors. It should be emphasized that solution A, eq. (29), providing the absolute maxima of $P_{\alpha\beta}$, can be and was derived directly from eqs. (15) and (22) without utilizing the standard extrema equations (24) - (25), i.e., *its derivation does not depend on the chosen set of variables* of $P_{\alpha\beta}$, which are varied to obtain the extrema conditions. Obviously, it can also be obtained for *a given set of variables* from eq. (23) and the corresponding extrema conditions of the type of (24) - (25).

The *absolute* minima of $P_{\alpha\beta}$ in the case under discussion are located on the intersections of the curves

$$min\ P_{\alpha\beta} = 0: \ \begin{cases} \sin\phi' = 0,\ or\ 2\phi' = 2\pi k',\ k' = 0, 1, ..., \\ \sin\phi'' = 0,\ or\ 2\phi'' = 2\pi k'',\ k'' = 0, 1, ... \end{cases} \tag{48}$$

Our results apply to the case when the first layer is vacuum $V' = 0$ (Fig. 2), because there is no principal difference between the neutrino oscillations in constant-density medium and in vacuum. If we assume the nonzero density layer to be the Earth mantle, maximal neutrino conversion (solution A) can take place provided neutrinos travel a distance of $\sim$ few$\times 10^3$ km in vacuum.

The above results are valid also, e.g., for neutrinos born in the central region of the Earth. Such neutrinos cross two layers - the Earth core and mantle, on the way to the Earth surface. If the dimensions of the region of neutrino production are small compared to the Earth core radius, the neutrino path lengths X' and X'' are fixed. The extrema condition with respect to the variable E is given in ref. [20]. It differs from the the extrema conditions (24) and (25). Accordingly, the solutions of the system (24) - (25) do not correspond, in general, to extrema in the variable E. More specifically, solutions B, C and D, corresponding to local maxima of $P_{\alpha\beta}$, $max\ P_{\alpha\beta} < 1$, in the variables ϕ' and ϕ'', do not provide local maxima in the variable E. Only solution A provides the absolute maxima of the neutrino transition probabilities of interest, $P_{\alpha\beta} = 1$, in any variable. Solution A is realized for ν_μ $(\nu_e) \rightarrow \nu_e$ $(\nu_{\mu;\tau})$, $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions of neutrinos born in the central region of the Earth, as Table 1 demonstrates.

4 Special Case of a Three - Layer Medium

Similar analysis can be performed for the transitions of neutrinos traversing three layers of constant density and chemical composition when the first and the third layers have the same density, chemical composition and width which differ, however, from those of the second layer. This corresponds to the physically important case of solar and atmospheric neutrinos crossing the Earth core on the way to detectors located near or on the Earth surface, the first and third layers being the Earth mantle, and the Earth core playing the role of the second layer.

We will use the same notations for the parameters of the first (third) and the second layers as in Section 2, and will assume that eq. (21) holds. The evolution

matrix in the case of interest is given by: $U = U'U''U'$. Accordingly, the parameters of U, eq. (10), can be expressed in terms of the parameters of the first (third) and the second layers as follows (see, e.g., [21]):

$$\left\{ \begin{array}{l} \cos\phi = \cos(2\phi')\cos\phi'' - \cos(2\theta''_m - 2\theta'_m)\sin(2\phi')\sin\phi'', \\ \mathbf{n}\sin\phi = \mathbf{n}'\left[\sin(2\phi')\cos\phi'' - (\mathbf{n}'\cdot\mathbf{n}'')(1-\cos(2\phi'))\sin\phi''\right] + \mathbf{n}''\sin\phi''. \end{array} \right. \tag{49}$$

An expression for the probability $P_{\alpha\beta}$ is given in [1] (see also [2]). The necessary conditions for a maximum of the probability $P_{\alpha\beta}$, obtained by varying the phases ϕ' and ϕ'', read

$$max\ P_{\alpha\beta}: \quad \left\{ \begin{array}{l} \sin(2\theta'_m)\ F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0, \\ F(2\theta''_m - 2\theta'_m, 2\theta'_m; \phi'' + \pi/2, 2\phi') = 0, \end{array} \right. \tag{50}$$

while the requirements for an *absolute* maximum have the form

$$max\ P_{\alpha\beta} = 1: \quad \left\{ \begin{array}{l} \cos\phi \equiv F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0, \\ n_3\sin\phi \equiv F(2\theta''_m - 2\theta'_m, 2\theta'_m + \pi/2; \phi'', 2\phi') = 0. \end{array} \right. \tag{51}$$

where the function $F(Y, Z; \varphi, \psi)$ is defined by eq. (26). Using the expression for $F(Y, Z; \varphi, \psi)$ we get the conditions (51) in explicit form:

$$max\ P_{\alpha\beta} = 1: \quad \left\{ \begin{array}{l} 2\cos(\phi')\cos\Phi - \cos\phi'' = 0, \\ \sin(\phi'')\cos(2\theta''_m) + 2\sin(\phi')\cos(2\theta'_m)\cos\Phi = 0, \end{array} \right. \tag{52}$$

where $\cos\Phi$ is given by eq. (22).

The *absolute* maxima of $P_{\alpha\beta}$,

$$case\ A: \ max\ P_{\alpha\beta} = 1, \tag{53}$$

are provided by the solutions of eq. (51) (or (52)), which can be found explicitly:

$$solution\ A: \quad \left\{ \begin{array}{l} \tan\phi' = \pm\sqrt{\dfrac{-\cos 2\theta''_m}{\cos(2\theta''_m - 4\theta'_m)}}, \\ \tan\phi'' = \pm\dfrac{\cos 2\theta'_m}{\sqrt{-\cos(2\theta''_m)\cos(2\theta''_m - 4\theta'_m)}}, \end{array} \right. \tag{54}$$

where the signs are correlated.

The probability $P_{\alpha\beta}$ ($P_{\bar{\alpha}\bar{\beta}}$) exhibits a system of of maxima which is similar to that in the two layer case. Under the conditions (8), (21) and if $V'_{\alpha\beta} > 0$ (i.e., for the ν_μ (ν_e) $\rightarrow \nu_e$ ($\nu_{\mu;\tau}$), $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions in the Earth), solutions (54) are realized in the region A (Figs. 3 - 5),

$$region\ A: \quad \left\{ \begin{array}{l} \cos(2\theta''_m) \leq 0, \\ \cos(2\theta''_m - 4\theta'_m) \geq 0. \end{array} \right. \tag{55}$$

On the line belonging to region A,

$$region\ B: \quad \cos 2\theta'_m = 0, \tag{56}$$

we have

$$case\ B:\ max\ P_{\alpha\beta} = \sin^2 2\theta'_m = 1, \tag{57}$$

provided

$$solution\ B: \begin{cases} \cos 2\phi' = 0,\ or\ 2\phi' = \frac{\pi}{2}(2k'+1),\ k' = 0,1,..., \\ \sin\phi'' = 0,\ or\ 2\phi'' = 2\pi k'',\ k'' = 0,1,... \end{cases} \tag{58}$$

Besides these *absolute* maxima, there exist two regions,

$$region\ C:\ \cos(2\theta''_m) \geq 0, \tag{59}$$

and

$$region\ D:\ \cos(2\theta''_m - 4\theta'_m) \leq 0, \tag{60}$$

with maxima

$$case\ C:\ max\ P_{\alpha\beta} = \sin^2 2\theta''_m, \tag{61}$$

and

$$case\ D:\ max\ P_{\alpha\beta} = \sin^2(2\theta''_m - 4\theta'_m), \tag{62}$$

which correspond to the solutions

$$solution\ C: \begin{cases} \sin\phi' = 0,\ or\ 2\phi' = 2\pi k',\ k' = 0,1,..., \\ \cos\phi'' = 0,\ or\ 2\phi'' = \pi(2k''+1),\ k'' = 0,1,..., \end{cases} \tag{63}$$

and

$$solution\ D: \begin{cases} \cos\phi' = 0,\ or\ 2\phi' = \pi(2k'+1),\ k' = 0,1,..., \\ \cos\phi'' = 0,\ or\ 2\phi'' = \pi(2k''+1),\ k'' = 0,1,..., \end{cases} \tag{64}$$

respectively. In contrast to the two-layer case, the region B is just a line, belonging actually to the region A, and on it only absolute maxima can be realized due to the MSW effect in the first (third) layer (the mantle in the case of the Earth). The case C corresponds to the MSW effect in the second layer, while the cases A and D correspond to the new resonance-like effect of constructive interference between transition amplitudes in the first (and third) and the second layers. Due to this effect the transition probability can reach its maximal value, $P_{\alpha\beta} = 1$, if the oscillation parameters $\sin^2(2\theta)$ and $\Delta m^2/E$ belong to the region A, eq. (55).

The *absolute* minima of the probability $P_{\alpha\beta}$ are determined by the equation

$$min\ P_{\alpha\beta} = 0 : F(2\theta''_m - 2\theta'_m, 2\theta'_m; \phi'', 2\phi') = 0, \tag{65}$$

while for $F(2\theta''_m - 2\theta'_m, 2\theta'_m; \phi'', 2\phi') \neq 0$, the necessary conditions for the minima are given by eq. (50).

We get the same system of maxima also in the $\nu_2 \to \nu_e$ transition probability, $P(\nu_2 \to \nu_e) \equiv P_{e2}$, ν_2 being the heavier of the two mass eigenstate neutrinos in vacuum, which can be used to account for the the Earth matter effects in the transitions of solar neutrinos traversing the Earth. The probability P_{e2} is given in the case of $\nu_e - \nu_{\mu(\tau)}$ mixing by the U_{e2} element, $P_{e2} = |U_{e2}|^2$, of the evolution matrix

$$U^{\odot} \equiv \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} U_{ee} & U_{e\mu} \\ U_{\mu e} & U_{\mu\mu} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv UO(\theta), \tag{66}$$

where $U = U'U''U'$. If $\nu_e - \nu_s$ mixing takes place, the index μ appearing in the above equation has to be replaced by s. The matrix $U^{\odot}$ can be written in the general form (10):

$$U^{\odot} = U'U''U'O = \cos\phi^{\odot} - i(\boldsymbol{\sigma n^{\odot}})\sin\phi^{\odot}. \tag{67}$$

It is not difficult to express the parameters of the evolution matrix (67), $\phi^{\odot}$ and $\boldsymbol{n^{\odot}}$, in terms of the parameters ϕ', $\boldsymbol{n'}$, ϕ'', $\boldsymbol{n''}$ and θ. The transition probability P_{e2} can be written in the form

$$P_{e2} = [F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta; \phi'', 2\phi')]^2 + \sin^2\theta\,[F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2)]^2\ . \tag{68}$$

Using eqs. (68) and (26) we get the necessary conditions for the *extrema* of P_{e2},

$$max\ P_{\alpha\beta}, min\ P_{\alpha\beta} : \left\{ \begin{array}{l} \sin(2\theta'_m)F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0, \\ F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta; \phi'' + \pi/2, 2\phi') = 0, \end{array} \right. \tag{69}$$

while from eqs. (15) and (16), utilizing eqs. (49) and (67), one obtains the requirements for *absolute* extrema of the probability P_{e2}:

$$max\ P_{e2} = 1 : \left\{ \begin{array}{l} \cos\phi^{\odot} \equiv F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0 \\ n_3^{\odot}\sin\phi^{\odot} \equiv F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta + \pi/2; \phi'', 2\phi') = 0, \end{array} \right. \tag{70}$$

$$min\ P_{e2} = 0 : \left\{ \begin{array}{l} n_1^{\odot}\sin\phi^{\odot} \equiv F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0 \\ n_2^{\odot}\sin\phi^{\odot} \equiv F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta; \phi'', 2\phi') = 0. \end{array} \right. \tag{71}$$

Conditions (70), for instance, have the following explicit form:

$$max\ P_{e2} = 1 : \left\{ \begin{array}{l} 2\cos(\phi')\cos\Phi - \cos\phi'' = 0, \\ \sin(\phi'')\cos(2\theta''_m - \theta) + 2\sin(\phi')\cos(2\theta'_m - \theta)\cos\Phi = 0. \end{array} \right. \tag{72}$$

These all are the same type of equations as (52) and we can easily find their solutions.

The absolute maxima of P_{e2},

$$case\ A^{\odot} : max\ P_{e2} = 1, \tag{73}$$

are reached for

$$solution\ A^{\odot} : \left\{ \begin{array}{l} \tan\phi' = \pm\sqrt{\dfrac{-\cos(2\theta''_m - \theta)}{\cos(2\theta''_m - 4\theta'_m + \theta)}}, \\ \tan\phi'' = \pm\dfrac{\cos(2\theta'_m - \theta)}{\sqrt{-\cos(2\theta''_m - \theta)\cos(2\theta''_m - 4\theta'_m + \theta)}}, \end{array} \right. \tag{74}$$

where the signs are correlated, in the region $A^{\odot}$ (Fig. 4),

$$region\ A^{\odot} : \left\{ \begin{array}{l} \cos(2\theta''_m - \theta) \leq 0 \\ \cos(2\theta''_m - 4\theta'_m + \theta) \geq 0. \end{array} \right. \tag{75}$$

There are the line

$$\textit{region } B^{\odot}: \;\; \cos(2\theta'_m - \theta) = 0, \tag{76}$$

belonging to A, and two bordering regions,

$$\textit{region } C^{\odot}: \;\; \cos(2\theta''_m - \theta) \geq 0, \tag{77}$$

and

$$\textit{region } D^{\odot}: \;\; \cos(2\theta''_m - 4\theta'_m + \theta) \leq 0, \tag{78}$$

where the solutions B, eq. (58); C, eq. (63); and D, eq. (64), are realized. These solutions correspond to the following maxima of P_{e2}:

$$\textit{case } B^{\odot}: \; \textit{max } P_{e2} = \sin^2(2\theta'_m - \theta) = 1, \tag{79}$$

$$\textit{case } C^{\odot}: \; \textit{max } P_{e2} = \sin^2(2\theta''_m - \theta), \tag{80}$$

and

$$\textit{case } D^{\odot}: \; \textit{max } P_{e2} = \sin^2(2\theta''_m - 4\theta'_m + \theta). \tag{81}$$

Solutions $D^{\odot}$ and D (eqs. ((60), (62), (64), (78), (81)) correspond to the NOLR discussed [7] in [1].

For the *absolute* minima of P_{e2} we get the following solutions:

$$\textit{min } P_{e2} = 0: \quad \begin{cases} \tan\phi' = \pm\sqrt{\dfrac{\sin(2\theta''_m - \theta)}{\sin(2\theta''_m - 4\theta'_m + \theta)}} \\ \tan\phi'' = \pm\dfrac{\sin(2\theta'_m - \theta)}{\sqrt{\sin(2\theta''_m - \theta)\sin(2\theta''_m - 4\theta'_m + \theta)}}, \end{cases} \tag{82}$$

where again the signs are correlated.

5 Transitions of Neutrinos Traversing the Earth Core

In the case of transitions in the Earth of (solar and atmospheric) neutrinos which pass through the Earth mantle, the core and the mantle again, the two lengths X' and X'' are not independent due to the spherical symmetry of the Earth, if both the Earth radius and the Earth core radius are fixed. We can choose as independent variables, for example, $\cos h$, h being the Nadir angle, and $y \equiv \Delta m^2/E$. The corresponding necessary conditions for the maxima of the probability $P_{\alpha\beta}$ read:

$$\frac{\mathrm{d}P_{\alpha\beta}}{\mathrm{d}\cos h} = 0: \qquad \frac{2R_{\oplus}}{X''}\left(\Delta E'' R_{\oplus}\cos h\,\frac{\mathrm{d}F}{\mathrm{d}\phi''} - 2\phi'\frac{\mathrm{d}F}{\mathrm{d}(2\phi')}\right) = 0, \tag{83}$$

$$\frac{\mathrm{d}P_{\alpha\beta}}{\mathrm{d}y} = 0: \qquad X'\,\frac{y - 2V'_{\alpha\beta}\cos(2\theta)}{4\Delta E'}\,\frac{\mathrm{d}F}{\mathrm{d}(2\phi')} + \frac{X''}{2}\,\frac{y - 2V''_{\alpha\beta}\cos(2\theta)}{4\Delta E''}\,\frac{\mathrm{d}F}{\mathrm{d}\phi''}$$

[7] Solutions B and C (eqs. (58) and (63)) were considered briefly in [1] as well.

$$+\frac{V'_{\alpha\beta}}{\Delta E'^2}\sin(2\theta)\sin\phi'\left[\cos(4\theta'_m-2\theta''_m)\sin\phi'\sin\phi''-\cos(2\theta'_m)\cos\phi'\cos\phi''\right]$$

$$-\frac{V''_{\alpha\beta}}{\Delta E''^2}\frac{\sin(2\theta)}{2}\sin\phi''\left[\cos(4\theta'_m-2\theta''_m)\sin^2\phi'+\cos(2\theta''_m)\cos^2\phi'\right]=0, \qquad (84)$$

where $F \equiv F(2\theta''_m - 2\theta'_m, 2\theta'_m; \phi'', 2\phi')$ is given by eq. (26), $\mathrm{d}F/\mathrm{d}\phi'' = F(2\theta''_m - 2\theta'_m, 2\theta'_m; \phi'' + \pi/2, 2\phi')$ and $\mathrm{d}F/\mathrm{d}(2\phi') = \sin 2\theta'_m F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2)$. It is clear, that, in general, the maxima of $P_{\alpha\beta}$ in the variables ϕ' and ϕ'' do not correspond to maxima in the variables $\cos h$ and $\Delta m^2/E$ (compare eqs. (50) with eqs. (83) - (84)). It is not difficult to check, however, that solutions A, eq. (54), for the *absolute* maxima, corresponding to a total neutrino conversion, $P_{\alpha\beta} = 1$, are solutions of the system of equations (83) - (84) as well. They give the absolute maxima of the neutrino transition probability $P_{\alpha\beta}$ in any variable. Equation (54) gives also the complete set of such solutions. Note that solution A coincides on the curves $\cos(2\theta'_m) = 0$, $\cos(2\theta''_m) = 0$ and $\cos(2\theta''_m - 4\theta'_m) = 0$ with B, C and D, respectively. However, solutions C and D, eqs. (63) and (64), for the *local* maxima with *max* $(P_{\alpha\beta}) < 1$ no longer correspond to extrema in the variables $\cos h$ and $\Delta m^2/E$. New solutions for the local maxima of $P_{\alpha\beta}$, which do not coincide with the solutions associated with phases equal to multiples of π, are possible. The same conclusions are valid for the solutions $A^{\odot}$, $C^{\odot}$ and $D^{\odot}$, eqs. (74), (63) and (64) (giving (80), (81)), for the absolute and local maxima of the probability P_{e2}. Our numerical studies confirm these conclusions.

The solutions A, eq. (54) ($A^{\odot}$, eq. (74)), providing a total neutrino conversion, $P_{\alpha\beta} = 1$ ($P_{e2} = 1$), exist for the Earth density profile and for all neutrino transitions of interest, $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$, $\nu_e \rightarrow \nu_{\mu(\tau)}$, $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$, and at small, intermediate and large mixing angles. They are realized for very different sets of values of the phases $2\phi'$ and $2\phi''$, which, are not multiples of π. In Tables 2 - 4 we give a rather complete list of these solutions for the transitions $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$, $\nu_e \rightarrow \nu_{\mu(\tau)}$, $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ of the Earth-core-crossing neutrinos, while Tables 5 - 6 illustrate the behavior of the probabilities $P_{\mu e}$ and P_{e2} in the region of these solutions at small $\sin^2 2\theta$ (see further). The maximal neutrino conversion solutions $A^{\odot}$ and A, eqs. (74) and (54), are responsible for the strong resonance-like enhancement of the $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$, $\nu_e \rightarrow \nu_{\mu(\tau)}$, $\nu_e \rightarrow \nu_s$, etc. transitions in the Earth of the Earth-core crossing solar and atmospheric neutrinos, discussed in [1, 2, 3, 4]. In [1] this enhancement was interpreted to be due to the neutrino oscillation length resonance - solutions D, eq. (64), for the neutrinos passing through the Earth core. At small mixing angles the values of the parameters at which the maximal neutrino conversion takes place for the $\nu_2 \rightarrow \nu_e \cong \nu_\mu \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\mu$ transitions are rather close to the values of the parameters for which the NOLR giving $P_{\alpha\beta} = 1$ occurs (Tables 2 and 4), while for the $\nu_2 \rightarrow \nu_e$ transitions caused by $\nu_e - \nu_s$ mixing and the $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions these values are very different [1, 2] (Tables 3 - 4). In both cases of transitions, however, only the maximal neutrino conversion mechanism is operative for the Earth-core-crossing neutrinos.

Let us consider the case of $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$ ($\nu_e \rightarrow \nu_\mu$) transitions at $\sin^2 2\theta \lesssim 0.10$ in somewhat greater detail. As it follows from Table 2, for, e.g., $h = 13^\circ$ we have

$P_{e2} = 1$ at $\sin^2 2\theta = 0.044$ and $\Delta m^2/E = 7.1 \times 10^{-7}$ eV^2/MeV, which corresponds to $2\phi' = 0.95\pi$ and $2\phi'' = 0.97\pi$. At the indicated point one finds also that $\sin^2(2\theta''_m - 4\theta'_m + \theta) = 1$, i.e., the NOLR solution D represents a very good approximation to solution A. Further, for any fixed $\sin^2 2\theta < 0.044$, P_{e2} has an absolute maximum in the variable $\Delta m^2/E$, which satisfies $max\ P_{e2} < 1$ (see Fig. 7 and, e.g., Fig. 1 in [1], Fig. 4b in [2]). When $\sin^2 2\theta$ increases from, say 10^{-3} to 0.044, these absolute maxima form a continuous curve - a "ridge", with respect to the variables $\sin^2 2\theta$ and $\Delta m^2/E$, which passes through the total neutrino conversion point $max\ P_{e2} = 1$. It was suggested in [1] on the basis of several test cases studied that the values of P_{e2} forming the "ridge" are given with a relatively good precision, by the NOLR, i.e., by eq. (3) (or (81)). These values have been calculated numerically and compared in Table 6 with the one predicted by eq. (3). We see from Table 6 that for $\sin^2 2\theta = 9.0 \times 10^{-3}$, eq. (3) reproduces the value of $max\ P_{e2}$ of interest with an error of 32%. For $h = 23^\circ$ and $\sin^2 2\theta = 10^{-2}$ the same error is 18%. The error increases with the decreasing of $\sin^2 2\theta$ and at $\sin^2 2\theta = 5.0 \times 10^{-3}$, for $h = 13^\circ;\ 23^\circ$ reaches 43%; 23%. Results of a similar analysis for the probability $P_{e\mu} = P_{\mu e}$ are collected in Table 5. As it follows from Table 5, for $h = 0^\circ;\ 23^\circ$ the NOLR, eq. (64), describes the values of $max\ P_{\mu e}$ on the corresponding "ridge" (see Fig. 6 and Figs. 5a - 5c in [2]) with an error which is 24%; 14% at $\sin^2 2\theta = 0.01$ and increases to 42%; 19% at $\sin^2 2\theta = 4 \times 10^{-3}$. More generally, Tables 5 and 6 show that for fixed $\sin^2 2\theta = (10^{-3} - 10^{-2})$ and fixed h, the NOLR describes the absolute maxima of P_{e2} and $P_{\mu e}$ in the variable $\Delta m^2/E$, $max\ P_{e2(\mu e)} < 1$, with an error which varies between 14% and 60%.

In some cases the total neutrino conversion solutions A and A° of interest occur for values of the parameters which are close to those for the solutions B, eq. (58), or C, eq. (63) (Tables 2 - 4). In all such cases the corresponding absolute maxima of the neutrino transition probabilities lie in the regions of the solutions A and A°, eqs. (55) and (75).

As Tables 2 - 4 and Figs. 6 - 9 indicate, for the Earth-core-crossing neutrinos, the new enhancement mechanism produces at $\sin^2 2\theta \lesssim 0.10$ one relatively broad (in $\sin^2 2\theta$, $\Delta m^2/E$ and h) resonance-like interference peak of total neutrino conversion in each of the probabilities P_{e2}, $P(\nu_\mu \rightarrow \nu_e)$, $P(\nu_e \rightarrow \nu_s)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$. The total conversion peaks of the different probabilities are located in the interval $0.03 \lesssim \sin^2 2\theta \lesssim 0.10$. They all take place for values of the resonance density $N^{res} = \Delta m^2 \cos 2\theta/(2E\sqrt{2}G_F)$, which differ from the number densities in the core and in the mantle $V'_{\alpha\beta}/(\sqrt{2}G_F)$ and $V''_{\alpha\beta}/(\sqrt{2}G_F)$, and lie between the latter two [1]: $V'_{\alpha\beta} < \sqrt{2}G_F N^{res} < V''_{\alpha\beta}$. For all the neutrino transitions considered, the corresponding interference peak is sufficiently wide in all variables, which makes the transitions observable in the region of the enhancement [8]: the relative width of the peak in the neutrino energy is $\Delta E/E_{max} \cong (0.3 - 0.5)$ and practically does not vary with $\sin^2 2\theta$ (see also [2, 4]), while in $\sin^2 2\theta$ it is $\sim (2.0 - 2.3)$; the absolute width in h for the peaks located at $h \lesssim 25^\circ$ is $\sim (25 - 30)^\circ$. The points on the "ridges" leading to these peaks at, say, $\sin^2 2\theta \lesssim 0.03$ represent for fixed h and $\sin^2 2\theta$ the dominating maxima in the variable $\Delta m^2/E$ of the probabilities of interest, with $max\ P_{\alpha,\beta} < 1$ (see Figs.

[8]Let us note that the peak is not symmetric in any of the variables.

6 - 9 and , e.g., Figs. 1 - 2 in [1] and Figs. 4b, 5a - 5c, 10a - 10b, 15a - 15c in [2]).

As Tables 2 - 4 and Figs. 6 - 9 show, the absolute maxima of total neutrino conversion, solution A, are present in the probabilities P_{e2}, $P(\nu_\mu \rightarrow \nu_e)$, $P(\nu_e \rightarrow \nu_s)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$, at large values of $\sin^2 2\theta \cong (0.50 - 1.0)$ as well. They are also present in certain transitions at $\sin^2 2\theta \cong (0.15 - 0.50)$. Although these resonance-like interference maxima are, in general, narrower than at small mixing angles (see Figs. 6 - 9), they are sufficiently wide and can have observable effects both at $\sin^2 2\theta \cong (0.50 - 1.0)$ and $\sin^2 2\theta \cong (0.15 - 0.50)$. For certain specific trajectories the values of the phases $2\phi'$ and $2\phi''$ at the points of total conversion are close to odd multiples of π and the NOLR solution D reproduces approximately the results corresponding to the exact solution A. This takes place i) for $P_{\mu e}$ at $h = 23^\circ$ and $\sin^2 2\theta = 0.848$ (Table 2), ii) for P_{e2} ($\nu_e - \nu_\mu$ mixing) at $h = 23^\circ$ and $\sin^2 2\theta = 0.941$ (Table 4), iii) for P_{es} at $h = 0^\circ$ and $\sin^2 2\theta = 0.999$ and at $h = 30^\circ$ and $\sin^2 2\theta = 0.985$ (Table 3), iv) for $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$ at $h = 30^\circ$ and $\sin^2 2\theta = 0.981$ (Table 3). This approximate equality of the two solutions is unstable with respect to the change of the parameters. Inspecting the Tables 2 - 4 one finds that for Nadir angles different from those indicated in i) - iv), the total neutrino conversion (solution A) takes place for values of the parameters at which the NOLR does not provide at all, or provides a poor description of the enhancement of interest. Note also that at $\sin^2 2\theta \sim 1$ one can have $\sin^2(2\theta''_m - 4\theta'_m) \cong 1$ at the points of total neutrino conversion, solution A, even for $2\phi'$ and $2\phi''$ which differ quite substantially from being odd multiples of π (as are the cases, e.g., of $max\ P_{\mu e} = 1$ at $h = 0^\circ$ and $\sin^2 2\theta = 0.994$, as well as of $max\ P_{es} = 1$ at $\sin^2 2\theta = 0.931$ and of $max\ P_{\bar{\mu}\bar{s}} = 1$ at $\sin^2 2\theta = 0.992$, for $h = 23^\circ$). In these cases the analytic expression for $P_{\alpha\beta}$ does not coincide with $\sin^2(2\theta''_m - 4\theta'_m)$.

5.1 Transitions of Atmospheric Neutrinos

$\nu_\mu \rightarrow \nu_e$, $\nu_e \rightarrow \nu_{\mu(\tau)}$. These can be, e.g., sub-dominant transitions of the atmospheric ν_μ, driven by the values of Δm^2 suggested by the Super-Kamiokande atmospheric neutrino data [11],

$$\Delta m^2 \cong (10^{-3} - 8 \times 10^{-3})\ \mathrm{eV}^2, \tag{85}$$

and by a relatively small mixing [1] (see also [2, 3, 4]). Such transitions should exist if three-flavour-neutrino (or four-neutrino) mixing takes place in vacuum, which is a very natural possibility in view of the present experimental evidences for oscillations of the flavour neutrinos. For the Earth-center-crossing neutrinos there are two solutions of the type A, providing a total neutrino conversion, $P_{\mu e} = P_{e\mu} = 1$, at small mixing angles (Table 2): at $\sin^2 2\theta = 0.034;\ 0.15$ and $\Delta m^2/E = 7.2;\ 4.8 \times 10^{-7}\ \mathrm{eV}^2/\mathrm{MeV}$. The first is reproduced approximately by the NOLR solution D while the second is a new type A solution. The above implies that for $\Delta m^2 = 10^{-3}\ \mathrm{eV}^2$, the total neutrino conversion occurs at $E = 1.4;\ 2.1$ GeV, while if $\Delta m^2 = 5 \times 10^{-3}\ \mathrm{eV}^2$, it takes place at $E = 7.0;\ 10.5$ GeV. Thus, if the value of Δm^2 lies in the region (85), the new effect of total neutrino conversion occurs for values of the energy E of the atmospheric ν_e and ν_μ which contribute either to the sub-GeV or to the multi-GeV samples of $e-$like and

$\mu-$like events in the Super-Kamiokande experiment. The implications are analogous to the one discussed in [1, 2, 3] in connection with the NOLR interpretation of the enhancement of $P_{\mu e}$ ($P_{e\mu}$). The new effect can produce an excess of e-like events in the region $-1 \leq \cos\theta_z \lesssim -0.8$, θ_z being the Zenith angle, in the multi-GeV (or a smaller one - in the sub-GeV) sample of atmospheric neutrino events, and should be responsible for at least part of the strong Zenith angle dependence, exhibited by the $\mu-$like multi-GeV (sub-GeV) Super-Kamiokande data.

The total neutrino conversion can take place at several values of $\Delta m^2/E$ at large mixing angles (Table 2), $\sin^2 2\theta \gtrsim 0.8$. This can have implications, in particular, for the interpretation of the Super-Kamiokande data on the sub-GeV e-like events [24, 25].

$\bar{\nu}_\mu \rightarrow \bar{\nu}_s$**.** A total neutrino conversion due to the solution A, eq. (54), takes place at $\Delta m^2/E \leq 10^{-6}$ eV^2/MeV both at small and large mixing angles. For the Earth-center-crossing neutrinos the absolute maxima, $P_{\bar{\mu}\bar{s}} = 1$, are realized at (Table 3) $\sin^2 2\theta = 0.07;\ 0.59;\ 0.78;\ 0.93;\ 0.999$ for $\Delta m^2/E = 4.1;\ 2.8;\ 9.7;\ 6.7;\ 3.4 \times 10^{-7}$ eV^2/MeV. For the statistically preferred value of $\Delta m^2 \cong 4 \times 10^{-3}$ eV^2 [11], this corresponds to $E = 9.8;\ 14.3;\ 4.12;\ 6.0;\ 11.8$ GeV, which is in the range of the multi-GeV $\mu-$like events, studied by the Super-Kamiokande experiment. The total neutrino conversion maxima are present at all $h \lesssim 30^\circ$, but at values of $\sin^2 2\theta$ and $\Delta m^2/E$, which vary with h. This variation (and the corresponding variation of $2\phi'$ and $2\phi''$) is quite significant at large $\sin^2 2\theta$.

We would like to add the following remark in connection with the enhancement of the transitions discussed. In the article by Q.Y. Liu and A. Yu. Smirnov, Nucl. Phys. B524 (1998) 505, it was noticed that in the case of muon (anti-)neutrinos crossing the Earth along the specific trajectory characterized by a Nadir angle $h \cong 28.4^\circ$, and for $\sin^2 2\theta \cong 1$ and $\Delta m^2/E \cong (1-2) \times 10^{-4}$ eV^2/GeV, the $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transition probability is enhanced [9]. The authors interpreted the enhancement as being due to the conditions $2\phi' = \pi$ and $2\phi'' = \pi$, which they claimed to be approximately satisfied and to produce *dominating local maxima* in the relevant transition probability $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$. Actually, for the values of the parameters of the examples chosen by Q.Y. Liu and A. Yu. Smirnov to illustrate this conclusion (Fig. 2 in their article) one has $2\phi' \cong (0.6-0.9)\pi$ and $2\phi'' \cong (1.2-1.5)\pi$. The indicated enhancement is due to the existence of a *nearby total neutrino conversion point* which for $h \cong 28.4^\circ$ is located at $\sin^2 2\theta \cong 0.94$ and $\Delta m^2/E \cong 2.4 \times 10^{-4}$ eV^2/GeV and at which $2\phi' \cong 0.9\pi$ and $2\phi'' \cong 1.1\pi$. This point lies in the region of solution A, but very close to the border line with solution D: the fact that the condition $\cos(2\theta''_m - 4\theta'_m) \cong 0$ is also fulfilled is crucial for having $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s) = 1$. As we have already noticed, for each given $h \lesssim 30^\circ$ there are several total neutrino conversion points at large values of $\sin^2 2\theta$, at which the phases $2\phi'$ and $2\phi''$ are not necessarily equal to π or to odd multiples of π (see Table 3). Thus, the explanation of the enhancement offered by the indicated authors is at best qualitative.

[9] The discussion in the indicated article, being rather qualitative is done for the $\nu_\mu \rightarrow \nu_s$ transitions as the authors presume that $\cos 2\theta \sim 0$ and that the matter term dominates over the energy-dependent term in the corresponding system of evolution equations.

$\nu_e \to \nu_s$**.** The atmospheric ν_e can undergo small mixing angle $\nu_e \to \nu_s$ transitions with Δm^2 from the interval (85). For the Earth-core-crossing neutrinos the total neutrino conversion (solution A) takes place at small mixing at $\Delta m^2/E \cong 3 \times 10^{-7}$ eV^2/MeV (Table 3), or for $E \cong (3 - 26)$ GeV if Δm^2 is given by eq. (85). Such transitions would lead to a reduction of the rate of the multi-GeV e-like events at $-1 \leq \cos\theta_z \lesssim -0.8$ in the Super-Kamiokande detector.

5.2 Transitions of Solar Neutrinos

In the case of the transitions of the Earth-core-crossing solar neutrinos the relevant probability is P_{e2}. The transitions can be generated by $\nu_e - \nu_{\mu(\tau)}$ or by $\nu_e - \nu_s$ mixing. The interference maxima of P_{e2}, corresponding to solution $A^{\odot}$ (eq. (74)), $P_{e2} = 1$, can take place only in the intervals $\Delta m^2/E \cong (1.9 - 8.2) \times 10^{-7}$ eV^2/MeV (Fig. 4) and $\Delta m^2/E \cong (0.9 - 3.5) \times 10^{-7}$ eV^2/MeV, respectively.

In the case of $\nu_e - \nu_{\mu(\tau)}$ mixing, the absolute maximum $P_{e2} = 1$ occurs both at small and large mixing angles (Table 4). At small mixing angles it takes place at $\sin^2 2\theta = 0.038;\ 0.044;\ 0.060;\ 0.077$ for $h = 0°;\ 13°;\ 23°;\ 30°$, respectively. These values of $\sin^2 2\theta$ lie outside the region of the MSW SMA solution. The change of P_{e2} in the variables $\sin^2 2\theta$ and $\Delta m^2/E$ along the "ridge" of local maxima leading to the "peak" $P_{e2} = 1$ at fixed h is illustrated in Table 6 (see also Figs. 1 - 2 in [1] and Fig. 4b in [2]). At $\sin^2 2\theta = 7 \times 10^{-3}$, for instance, we have at the "ridge" $P_{e2} = 0.36;\ 0.28$ for $h = 13°;\ 23°$. These local maxima dominate in P_{e2}: the local maxima associated with the MSW effect in the core (mantle) are by the factor $\sim (2.5 - 4.0)$ ($\sim (3 - 7)$) smaller [1]. For the values of $\Delta m^2 \cong (4 - 9) \times 10^{-6}$ eV^2 of the MSW SMA solution region, the dominating local maxima take place at values of the solar neutrino energy from the interval $E \cong (5.6 - 13.8)$ MeV [1], to which the Super-Kamiokande, SNO and ICARUS experiments are sensitive.

At large mixing angles, $\sin^2 2\theta \gtrsim 0.5$, we have $P_{e2} = 1$ at values of $\sin^2 2\theta$ which fall in the region of the large mixing angle (LMA) solution of the solar neutrino problem (Table 4). For the LMA solution values of $\Delta m^2 \cong (1 - 10) \times 10^{-5}$ eV^2 the absolute maxima $P_{e2} = 1$ occur at $E \gtrsim 12.5$ MeV (see also [2]) and typically at even larger values of the neutrino energy. Nevertheless, their presence has an influence on the transitions even at smaller E and can enhance them somewhat (Fig. 7, see also [5]).

The solar ν_e can undergo SMA MSW transitions into ν_s for $\Delta m^2 \cong (3.0 - 8.0) \times 10^{-6}$ eV^2 (see, e.g., [23, 10]). Taking into account the interval of values of $\Delta m^2/E$ for which we can have total neutrino conversion, one finds that solution A can be realized only for $E \gtrsim 9$ MeV.

The implications of the new enhancement effect for the interpretation of the data of the solar neutrino experiments have been discussed in [1] and in much greater detail in [5, 6, 8, 9]. Let us just mention here that due to this enhancement it would be possible to probe at least part of the $\Delta m^2 - \sin^2 2\theta$ region of the SMA MSW $\nu_e \to \nu_{\mu(\tau)}$ transition solution of the solar neutrino problem by performing selective D-N effect measurements [5, 8, 9]. The Super-Kamiokande collaboration is already exploiting successfully these results [10].

Recently the Super-Kamiokande Collaboration has published for the first time

data on the event rate produced only by solar neutrinos which cross the Earth core on the way to the detector [26]: in contrast to their previous night bin N5 data, the new *Core* data are not "contaminated" by contributions from neutrinos which cross the Earth mantle but do not cross the Earth core. Due to the new enhancement effect, the Super-Kamiokande *Core* data and the corresponding value of the *Core* D-N asymmetry are particularly useful in performing effective tests the MSW SMA $\nu_e \rightarrow \nu_{\mu(\tau)}$ solution, as was suggested in [5] and the results of a simplified analysis show [9]. More specifically, at 2 s.d. a large subregion of the MSW SMA $\nu_e \rightarrow \nu_{\mu(\tau)}$ solution region is incompatible with the D-N effect data (including the *Core* one), while at 3 s.d. the data on the *Core* D-N asymmetry *alone* excludes a non-negligible subregion of the indicated solution region (see Fig. 1c in [9]). The results obtained in [9] indicate also that the data on the *Core* and *Night* D-N asymmetries can be used to perform rather effective tests of the MSW LMA $\nu_e \rightarrow \nu_{\mu(\tau)}$ solution as well.

The consequences of the new enhancement effect for the interpretation of the solar neutrino data are less significant in the case of the MSW $\nu_e \rightarrow \nu_s$ solution of the solar neutrino problem [6]. Nevertheless, using the Super-Kamiokande data on the *Core* D-N asymmetry [26] one can probe and constrain also the MSW $\nu_e \rightarrow \nu_s$ "conservative" solution region [9].

Similarly, the future data on the *Core* and *Night* D-N asymmetries in, e.g., the charged current event rate in the SNO detector [27] can be used to perform equally effective tests of the MSW solutions of the solar neutrino problem [8].

6 Conclusions

We have investigated the extrema of probabilities of the two-neutrino transitions ν_μ (ν_e) $\rightarrow \nu_e$ ($\nu_{\mu;\tau}$), $\nu_2 \rightarrow \nu_e$, $\nu_e \rightarrow \nu_s$, etc. in a medium of nonperiodic density distribution, consisting of i) two layers of different constant densities, and ii) three layers of constant density with the first and the third layers having identical densities and widths which differ from those of the second layer. The first case would correspond, e.g., to neutrinos produced in the central region of the Earth and traversing both the Earth core and mantle on the way to the Earth surface. The second corresponds to, e.g., the Earth-core-crossing solar and atmospheric neutrinos which pass through the Earth mantle, the core and the mantle again on the way to the detectors. For both media considered we have found that in addition to the local maxima corresponding to the MSW effect and the NOLR (neutrino oscillation length resonance), there exist absolute maxima caused by a new effect of enhancement and corresponding to a total neutrino conversion, $P_{\alpha\beta} = 1$. The conditions for existence and the complete set of the new absolute maxima were derived. These conditions differ from the conditions of enhancement of the probability of transitions of neutrinos propagating in a medium with periodically varying density, discussed in [18]. The absolute maxima associated with the new conditions of total neutrino conversion are absolute maxima corresponding to $P_{\alpha\beta} = 1$ in any independent variable characterizing the neutrino transitions: the neutrino energy, the width of one of the layers, etc. It was shown that the new effect of total neutrino conversion takes place, in particular, in

the transitions ν_μ $(\nu_e) \to \nu_e$ $(\nu_{\mu;\tau})$, $\nu_2 \to \nu_e$, $\nu_e \to \nu_s$ and $\bar{\nu}_\mu \to \bar{\nu}_s$ in the Earth of the Earth-core-crossing solar and atmospheric neutrinos. The strong resonance-like enhancement of these transitions discussed in [1, 2] is due to the new effect. This enhancement was associated in [1] with the NOLR and we show that in certain cases, like the $\nu_2 \to \nu_e \cong \nu_\mu \to \nu_e$ and $\nu_e \to \nu_{\mu(\tau)}$ transitions at small mixing angles, the NOLR describes approximately the enhancement. However, in a number of cases the NOLR interpretation of the enhancement fails completely. A "catalog" of the most relevant absolute maxima corresponding to a total neutrino conversion, was given for the transitions indicated above. We have shown that the NOLR and the newly found enhancement effect are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. Thus, the maxima they produce in the neutrino transition probabilities are of interference nature.

Our analyses were done under the simplest assumption of existence of two-neutrino mixing (with nonzero-mass neutrinos) in vacuum: $\nu_e - \nu_{\mu(\tau)}$ or $\nu_e - \nu_s$, or else $\nu_\mu - \nu_s$. However, in many cases of practical interest the probabilities of the relevant three- (or four-) neutrino mixing transitions in the Earth reduce effectively to two-neutrino transition probabilities for which our results are valid. Thus, our results have a wider application than just in the case of two-neutrino mixing.

We have discussed also briefly the phenomenological implications of the new effect of total neutrino conversion in the transitions of the solar and atmospheric neutrinos traversing the Earth core. The effect leads to a strong resonance-like enhancement of the $\nu_2 \to \nu_e$ transitions of solar neutrinos if the solar neutrino problem is due to MSW small mixing angle $\nu_e \to \nu_\mu$ transitions in the Sun. The same effect should be operative also in the $\nu_\mu \to \nu_e$ $(\nu_e \to \nu_{\mu(\tau)})$ transitions of the atmospheric neutrinos crossing the Earth core if the atmospheric ν_μ and $\bar{\nu}_\mu$ indeed take part in large mixing vacuum $\nu_\mu \leftrightarrow \nu_\tau$, $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_\tau$ oscillations with $\Delta m^2 \sim (1-8) \times 10^{-3})$ eV^2, as is strongly suggested by the Super-Kamiokande atmospheric neutrino data, and if all three flavour neutrinos are mixed in vacuum. The existence of three-flavour-neutrino mixing in vacuum is a very natural possibility in view of the present experimental evidences for oscillations of the flavour neutrinos. In both cases the new effect of total neutrino conversion produces a strong enhancement of the corresponding transition probabilities, making the transitions observable even at rather small mixing angles (see also [1, 2]). Actually, the effect may have already manifested itself producing at least part of the strong Zenith angle dependence in the Super-Kamiokande multi-GeV $\mu-$like data [1, 2, 4]. The new effect should also be present in the $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_s$ transitions of the atmospheric multi-GeV $\bar{\nu}_\mu$'s both at small, intermediate and large mixing angles, if the atmospheric neutrinos undergo such transitions.

As the results obtained in refs. [1, 2, 5, 8] and in the present study indicate, it is not excluded that some of the current or future high statistics solar and/or atmospheric neutrino experiments will be able to observe directly the resonance-like enhancement of the neutrino transitions due to the effect of total neutrino conversion. Solar neutrino experiments located at lower geographical latitudes than the existing ones or those under construction (see the article by J.M. Gelb et al. quoted in [13]) would be better suited for this purpose. An atmospheric neutrino detector capable of reconstructing with relatively good precision, e.g.,, the energy and the direction of

the momentum of the incident neutrino in each neutrino-induced event, would allow to study in detail the new effect of total neutrino conversion in the oscillations of the atmospheric neutrinos crossing the Earth.

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Table 1: The values of $\Delta m^2/E \leq 20 \times 10^{-7}$ eV^2/MeV and of $\sin^2 2\theta$, at which the absolute maxima of $P_{e\mu} = P_{\mu e}$, P_{es} and $P_{\bar{\mu}\bar{s}}$, corresponding to a total neutrino conversion, $P_{\alpha\beta} = 1$ (solutions A, eq. (29)), take place for neutrinos traveling from the center of the Earth to the Earth surface. The values of $2\phi'$, $2\phi''$ (in units of π), $\sin^2 2\theta'_m$, $\sin^2 2\theta''_m$ and $\sin^2(2\theta''_m - 2\theta'_m)$ at the absolute maxima are also given.

$\frac{\Delta m^2}{E}$ $[10^{-7}\ \frac{eV^2}{MeV}]$	$\sin^2 2\theta$	$2\phi'/\pi$	$2\phi''/\pi$	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 2\theta'_m)$
$P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_{\mu(\tau)})$						
6.611	.138	.867	.888	.434	.603	.999
5.825	.651	1.094	1.866	.999	.501	.521
11.736	.649	2.356	2.679	.872	.985	.221
18.970	.828	4.153	4.853	.935	1.000	.072
13.589	.938	3.063	3.933	1.000	.885	.117
$P(\nu_e \to \nu_s)$						
3.355	.359	.531	.608	.774	.862	.590
6.073	.883	1.333	1.652	.993	.943	.103
10.181	.904	2.281	2.723	.975	.999	.037
13.663	.984	3.155	3.844	1.000	.983	.019
17.809	.962	4.089	4.912	.989	1.000	.012
$P(\bar{\nu}_\mu \to \bar{\nu}_s)$						
4.006	.275	.625	.684	.612	.745	.870
5.966	.850	1.287	1.679	.989	.848	.234
10.441	.849	2.303	2.709	.945	.996	.086
13.589	.981	3.133	3.864	1.000	.959	.045
18.002	.936	4.106	4.896	.974	1.000	.027

Table 2: The values of $\Delta m^2/E \leq 20 \times 10^{-7}$ eV^2/MeV and of $\sin^2 2\theta$, at which the absolute maxima of $P_{e\mu} = P_{\mu e}$, corresponding to a total neutrino conversion, $P_{e\mu} = 1$ (solution A, eq. (54)), take place for neutrinos crossing the Earth along trajectories passing through the Earth core: $h = 0°;\ 13°;\ 23°;\ 30°$. The values of $2\phi'$, $2\phi''$ (in units of π), $\sin^2 2\theta'_m$, $\sin^2 2\theta''_m$ and $\sin^2(2\theta''_m - 4\theta'_m)$ at the absolute maxima are also given.

$P(\nu_e \to \nu_{\mu(\tau)}) = P(\nu_\mu \to \nu_e)$							
h°	$\frac{\Delta m^2}{E}$ $[10^{-7}\ \frac{eV^2}{MeV}]$	$\sin^2 2\theta$	$2\phi'/\pi$	$2\phi''/\pi$	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 4\theta'_m)$
0.	7.201	.034	.923	.949	.111	.614	1.000
0.	4.821	.154	.509	2.334	.749	.208	.296
0.	11.256	.547	2.174	4.693	.795	.996	.285
0.	3.407	.568	.65	3.612	.846	.160	.881
0.	18.113	.803	3.933	9.129	.923	1.000	.710
0.	11.292	.841	2.425	6.167	.987	.891	.716
0.	19.307	.903	4.314	10.382	.979	.987	.845
0.	4.587	.948	1.167	4.705	.792	.285	.996
0.	19.862	.994	4.628	11.738	.992	.900	.981
0.	11.373	.994	2.702	7.577	.954	.708	.981
13.	7.024	.039	.931	.950	.131	.553	1.000
13.	4.537	.169	.496	2.258	.845	.179	.133
13.	10.071	.379	1.887	3.180	.645	.999	.068
13.	10.954	.626	2.272	4.503	.869	.973	.383
13.	3.267	.654	.732	3.456	.777	.153	.966
13.	6.736	.802	1.477	4.059	.997	.580	.688
13.	15.802	.874	3.653	7.680	.977	.972	.791
13.	18.249	.843	4.205	8.603	.949	.997	.762
13.	10.741	.927	2.529	5.936	.998	.797	.859
13.	18.770	.979	4.542	9.914	.999	.922	.954
23.	6.460	.051	.910	.922	.195	.387	1.000
23.	3.812	.224	.499	2.008	1.000	.125	.111
23.	10.101	.449	2.194	2.675	.726	.991	.134
23.	15.419	.744	3.875	5.236	.899	.999	.612
23.	3.417	.848	1.031	2.948	.711	.177	1.000
23.	9.655	.875	2.495	4.011	1.000	.786	.772
23.	16.565	.897	4.368	6.275	.984	.969	.828
23.	20.010	.979	5.470	8.061	.999	.935	.955
30.	5.376	.065	.765	.738	.352	.180	.982
30.	9.038	.199	2.062	.920	.418	1.000	.030
30.	2.908	.437	.742	1.427	.735	.095	.969
30.	14.060	.707	4.159	2.702	.885	.998	.548
30.	8.290	.825	2.491	2.021	1.000	.723	.691
30.	19.839	.913	6.326	4.357	.982	.986	.860
30.	17.123	.952	5.528	3.941	.999	.936	.909
30.	13.989	.997	4.694	3.597	.969	.785	.984
30.	8.909	1.000	3.139	2.741	.882	.551	.999

Table 3: The same as in Table 2 for the probabilities P_{es} and $P_{\bar{\mu}\bar{s}}$ and for $\Delta m^2/E \leq 10 \times 10^{-7}$ eV^2/MeV.

$P(\nu_e \to \nu_s)$							
h°	$\frac{\Delta m^2}{E}$ $[10^{-7}$ $\frac{eV^2}{MeV}]$	$\sin^2 2\theta$	$2\phi'/\pi$	$2\phi''/\pi$	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 4\theta'_m)$
0.	3.202	.101	.409	.626	.334	.834	.473
0.	2.715	.784	.565	1.843	.981	.538	.797
0.	9.334	.863	2.064	4.876	.956	1.000	.830
0.	6.997	.920	1.564	3.867	.997	.952	.892
0.	4.101	.999	1.013	2.976	.887	.599	1.000
13.	3.144	.110	.420	.599	.370	.799	.444
13.	2.660	.833	.606	1.751	.957	.505	.873
13.	7.456	.859	1.714	3.560	.971	.990	.816
13.	9.450	.893	2.213	4.581	.973	.998	.865
13.	4.324	.998	1.111	2.789	.903	.630	1.000
23.	2.965	.139	.441	.523	.480	.691	.346
23.	8.045	.817	2.072	2.861	.940	1.000	.768
23.	2.605	.931	.739	1.482	.883	.445	.980
23.	5.160	.958	1.403	2.206	.989	.812	.946
23.	9.433	.987	2.593	3.810	.997	.937	.981
30.	2.611	.196	.454	.372	.710	.503	.117
30.	6.358	.720	1.878	1.232	.903	.999	.628
30.	7.854	.978	2.573	1.850	.997	.919	.968
30.	2.835	.985	1.020	.964	.834	.444	1.000
30.	9.994	.996	3.317	2.371	.990	.923	.993
$P(\bar{\nu}_\mu \to \bar{\nu}_s)$							
0.	4.098	.070	.574	.719	.192	.713	.887
0.	2.771	.593	.497	2.011	1.000	.356	.329
0.	9.686	.782	2.096	4.817	.905	1.000	.641
0.	6.748	.926	1.511	3.976	1.000	.843	.820
0.	3.428	.999	.886	3.205	.811	.361	.999
13.	4.000	.077	.582	.701	.218	.662	.881
13.	2.635	.656	.524	1.926	.992	.323	.497
13.	7.750	.768	1.730	3.511	.920	.983	.581
13.	9.698	.834	2.231	4.554	.941	.994	.711
23.	3.700	.101	.587	.642	.306	.520	.842
23.	8.415	.715	2.107	2.800	.869	1.000	.525
23.	2.382	.825	.632	1.667	.895	.261	.865
23.	9.327	.983	2.558	3.879	.997	.880	.942
23.	4.528	.992	1.299	2.436	.919	.531	.969
30.	3.114	.150	.548	.488	.530	.317	.625
30.	6.798	.551	1.913	1.152	.762	1.000	.260
30.	7.795	.970	2.544	1.907	.997	.845	.911
30.	2.450	.981	.927	1.127	.751	.242	1.000
30.	9.742	1.000	3.274	2.463	.969	.815	.992

Table 4: The same as in Table 2 for the probability P_{e2} (solution $A^{\odot}$, eq. (74)) in the cases of $\nu_e - \nu_{\mu(\tau)}$ and $\nu_e - \nu_s$ mixing, and for $\Delta m^2/E \leq 10 \times 10^{-7}$ eV^2/MeV.

$P(\nu_2 \to \nu_e)$ ($\nu_e - \nu_{\mu(\tau)}$ mixing)							
h°	$\frac{\Delta m^2}{E}$ $[10^{-7}\ \frac{eV^2}{MeV}]$	$\sin^2 2\theta$	$2\phi'/\pi$	$2\phi''/\pi$	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 4\theta'_m + \theta)$
0.	7.275	.038	.944	.971	.122	.671	1.000
0.	4.684	.168	.498	2.423	.809	.199	.370
0.	2.724	.574	.602	3.755	.637	.096	.854
13.	7.100	.044	.953	.974	.145	.611	1.000
13.	4.370	.183	.480	2.345	.902	.167	.178
13.	2.434	.671	.680	3.599	.513	.081	.953
13.	8.025	.943	1.937	5.111	.967	.611	.314
23.	6.507	.060	.931	.951	.223	.434	1.000
23.	3.558	.235	.480	2.088	.983	.106	.110
23.	7.915	.751	1.902	3.160	.992	.730	.099
23.	5.939	.956	1.705	3.545	.885	.416	.428
23.	1.920	.941	.945	3.103	.296	.056	1.000
30.	5.314	.077	.761	.767	.408	.192	.984
30.	7.800	.385	1.847	1.196	.752	.853	.040
30.	2.425	.437	.730	1.486	.528	.061	.972
30.	7.183	.489	1.765	1.334	.886	.738	.002
$P(\nu_2 \to \nu_e)$ ($\nu_e - \nu_s$ mixing)							
0.	3.086	.122	.391	.692	.410	.765	.581
0.	1.646	.973	.488	2.026	.599	.203	.823
13.	3.017	.134	.400	.666	.457	.724	.542
13.	1.493	1.000	.531	1.934	.472	.156	.904
23.	2.802	.173	.415	.589	.601	.605	.405
30.	2.373	.251	.421	.425	.873	.407	.087

Table 5: Values of $P_{e\mu} = P_{\mu e}$ along the "ridge" of local maxima in the variables $\Delta m^2/E$ (in units of 10^{-7} eV^2/MeV) and $\sin^2 2\theta \leq 0.05$ for fixed h, when $\sin^2 2\theta$ decreases from the value at which the absolute maximum of $P_{e\mu} = 1$ (i.e., total neutrino conversion) takes place. The results correspond to Earth-core-crossing neutrinos: $h = 0°$; $13°$; $23°$. The values of $\sin^2 2\theta'_m$, $\sin^2 2\theta''_m$, $\sin^2(2\theta''_m - 4\theta'_m)$ and $2\phi'$, $2\phi''$ (in units of π), at the local maxima are also given.

h°	$\sin^2 2\theta$	$\frac{\Delta m^2}{E}$	$P_{\mu e}$	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 4\theta'_m)$	$2\phi'/\pi$	$2\phi''/\pi$
0.	.034	7.201	1.000	.111	.614	1.000	.923	.949
0.	.024	7.194	.951	.082	.549	.976	.915	.855
0.	.020	7.191	.890	.069	.509	.941	.912	.810
0.	.015	7.187	.760	.051	.439	.854	.907	.746
0.	.010	7.183	.562	.033	.339	.695	.902	.677
0.	.008	7.182	.500	.028	.307	.637	.901	.658
0.	.007	7.181	.432	.023	.271	.570	.900	.639
0.	.005	7.180	.359	.019	.231	.491	.898	.620
0.	.004	7.179	.279	.014	.185	.398	.897	.599
0.	.003	7.178	.193	.009	.132	.289	.896	.579
0.	.001	7.177	.100	.005	.071	.158	.895	.557
13.	.039	7.024	1.000	.131	.553	1.000	.931	.950
13.	.030	7.018	.967	.102	.498	.982	.923	.878
13.	.025	7.015	.916	.087	.460	.950	.918	.837
13.	.020	7.012	.836	.071	.415	.895	.914	.795
13.	.014	7.008	.675	.050	.336	.768	.908	.735
13.	.009	7.005	.504	.034	.256	.613	.904	.686
13.	.006	7.003	.362	.023	.189	.466	.901	.652
13.	.005	7.002	.282	.017	.150	.374	.900	.634
13.	.003	7.001	.195	.011	.106	.268	.898	.616
13.	.002	7.000	.101	.006	.056	.145	.897	.597
23.	.051	6.460	1.000	.195	.387	1.000	.910	.922
23.	.041	6.455	.976	.160	.343	.985	.900	.876
23.	.031	6.450	.891	.123	.288	.919	.889	.828
23.	.024	6.447	.800	.100	.248	.843	.883	.798
23.	.020	6.445	.720	.084	.217	.772	.878	.778
23.	.016	6.443	.621	.068	.184	.680	.874	.756
23.	.010	6.440	.435	.043	.125	.494	.867	.724
23.	.008	6.439	.361	.035	.103	.416	.865	.712
23.	.006	6.438	.281	.026	.080	.329	.863	.701
23.	.004	6.437	.194	.017	.055	.231	.860	.689
23.	.002	6.436	.101	.009	.029	.122	.858	.678

Table 6: The same as in Table 5 for the probability P_{e2} ($\nu_e - \nu_{\mu(\tau)}$ mixing).

h°	$\sin^2 2\theta$	$\frac{\Delta m^2}{E}$	P_{e2}	$\sin^2 2\theta'_m$	$\sin^2 2\theta''_m$	$\sin^2(2\theta''_m - 4\theta'_m + \theta)$	$2\phi'/\pi$	$2\phi''/\pi$
13.	.044	7.100	1.000	.145	.611	1.000	.953	.974
13.	.030	7.092	.934	.102	.536	.965	.941	.859
13.	.025	7.089	.864	.084	.494	.923	.936	.811
13.	.019	7.086	.762	.067	.441	.855	.931	.761
13.	.011	7.081	.503	.037	.311	.640	.923	.669
13.	.009	7.080	.435	.031	.275	.574	.922	.649
13.	.007	7.079	.361	.025	.235	.495	.920	.628
13.	.005	7.078	.281	.019	.188	.403	.919	.607
13.	.004	7.077	.195	.013	.135	.292	.917	.585
13	.002	7.076	.101	.006	.073	.160	.915	.563
23.	.060	6.507	1.000	.223	.434	1.000	.931	.951
23.	.024	6.492	.724	.097	.256	.782	.894	.782
23.	.012	6.487	.439	.050	.152	.509	.882	.718
23.	.010	6.486	.365	.040	.126	.431	.879	.704
23.	.007	6.485	.284	.030	.098	.343	.877	.690
23.	.005	6.484	.197	.020	.068	.242	.874	.676
23.	.002	6.483	.102	.010	.036	.129	.872	.662

FIGURE CAPTIONS

Figure 1. The regions of the four different solutions A, B, C and D (eqs. (29), (31), (37) and (45)) for the maxima of the transition probability $P_{e\mu} = P_{\mu e}$ in a two-layer medium. The two different layers correspond to the core and the mantle of the Earth [16, 17].

Figure 2. The same as in figure 1 for the case vacuum - Earth mantle.

Figure 3. The regions of the three different solutions A, C and D (eqs. (54), (63) and (64)) for the maxima of the transition probability $P_{e\mu} = P_{\mu e}$ in a three-layer medium. The three different layers correspond to the mantle-core-mantle of the Earth.

Figure 4. The same as in figure 3 for the transition probability P_{e2} in the case of $\nu_e - \nu_{\mu(\tau)}$ mixing.

Figure 5. The same as in figure 3 for the transition probability $P_{\bar{\mu}\bar{s}} \equiv P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$.

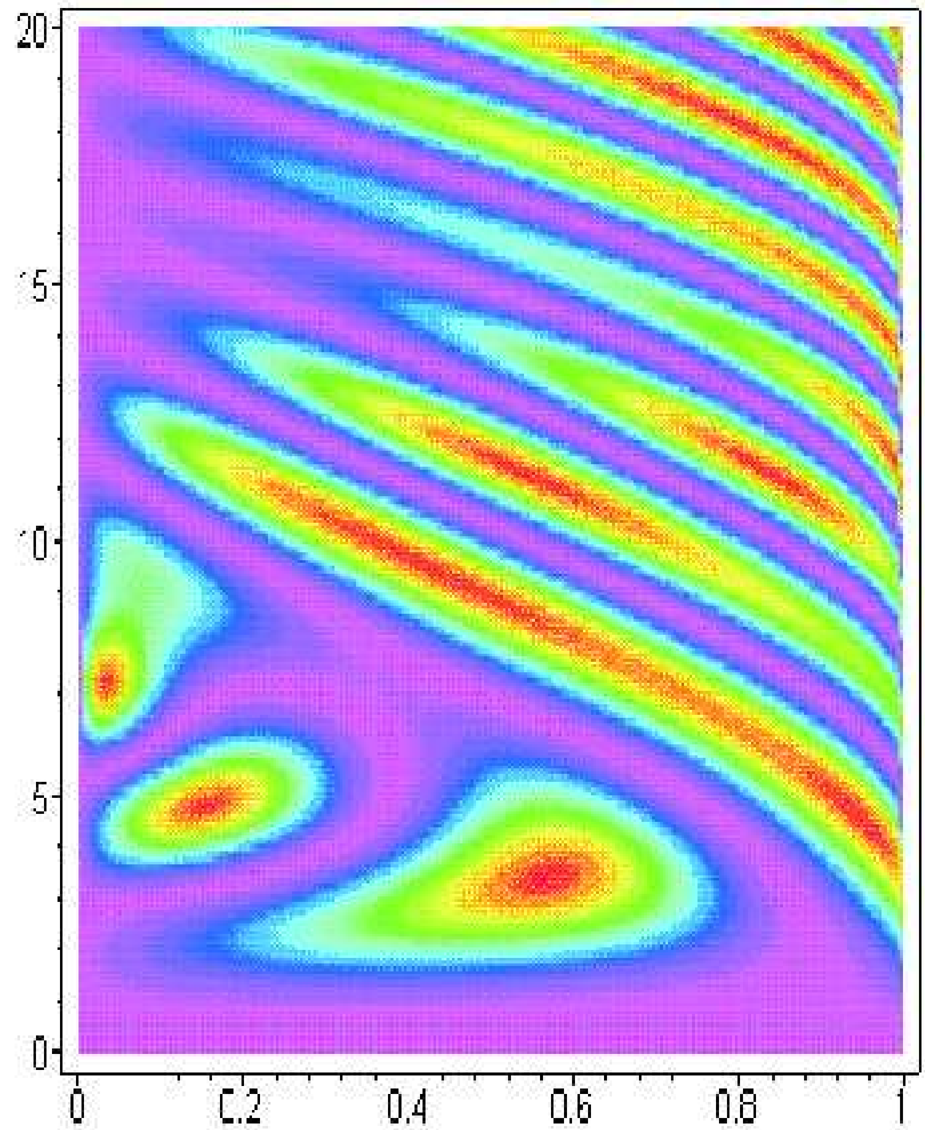
Figure 6. The probability $P_{e\mu} = P_{\mu e}$ for the Earth-center-crossing (atmospheric) neutrinos ($h = 0^\circ$), as a function of $\sin^2 2\theta$ (horizontal axis) and $\Delta m^2/E$ [10^{-7} eV^2/MeV] (vertical axis). The ten different colors correspond to values of $P_{e\mu}$ in the intervals: 0.0 - 0.1 (violet); 0.1 - 0.2 (dark blue); ...; 0.9 - 1.0 (dark red). The points of total neutrino conversion (in the dark red regions), $P_{e\mu} = 1$, correspond to solution A, eq. (54), for the Earth-core-crossing neutrinos.

Figure 7. The same as in figure 6 for the probability P_{e2} in the case of $\nu_e - \nu_{\mu(\tau)}$

mixing, and for (solar) neutrinos crossing the Earth (core) along the trajectory with $h = 13°$. The points of total neutrino conversion (in the dark red regions), $P_{e2} = 1$, correspond to solution A, eq. (74).

Figure 8. The same as in figure 6 for the probability $P_{\bar{\mu}\bar{s}}$ and for neutrinos crossing the Earth (core) along the trajectory with $h = 23°$. The points of total neutrino conversion (in the dark red regions), $P_{\bar{\mu}\bar{s}} = 1$, correspond to solution A, eq. (54).

Figure 9. The same as in figure 6 for the probability P_{es} and for neutrinos crossing the Earth (core) along the trajectory with $h = 0°$. The points of total neutrino conversion (in the dark red regions), $P_{es} = 1$, correspond to solution A, eq. (54).



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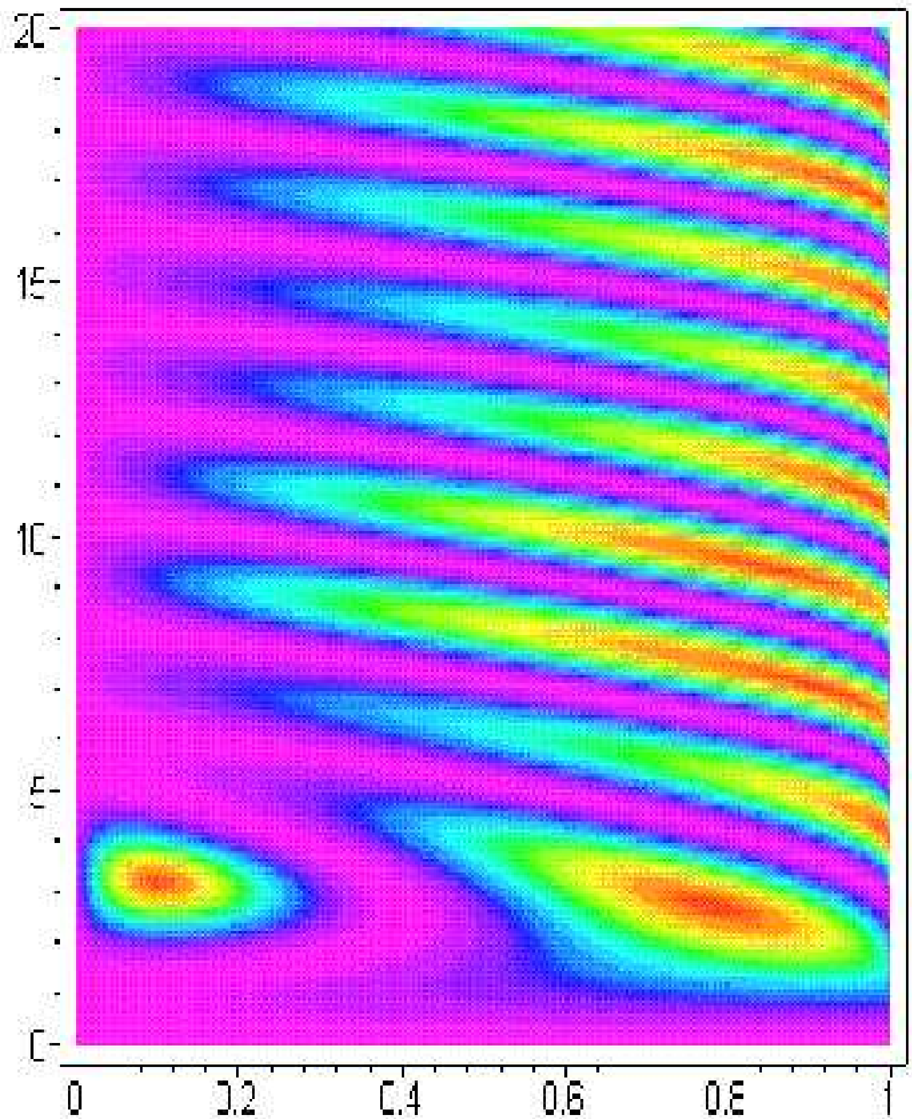
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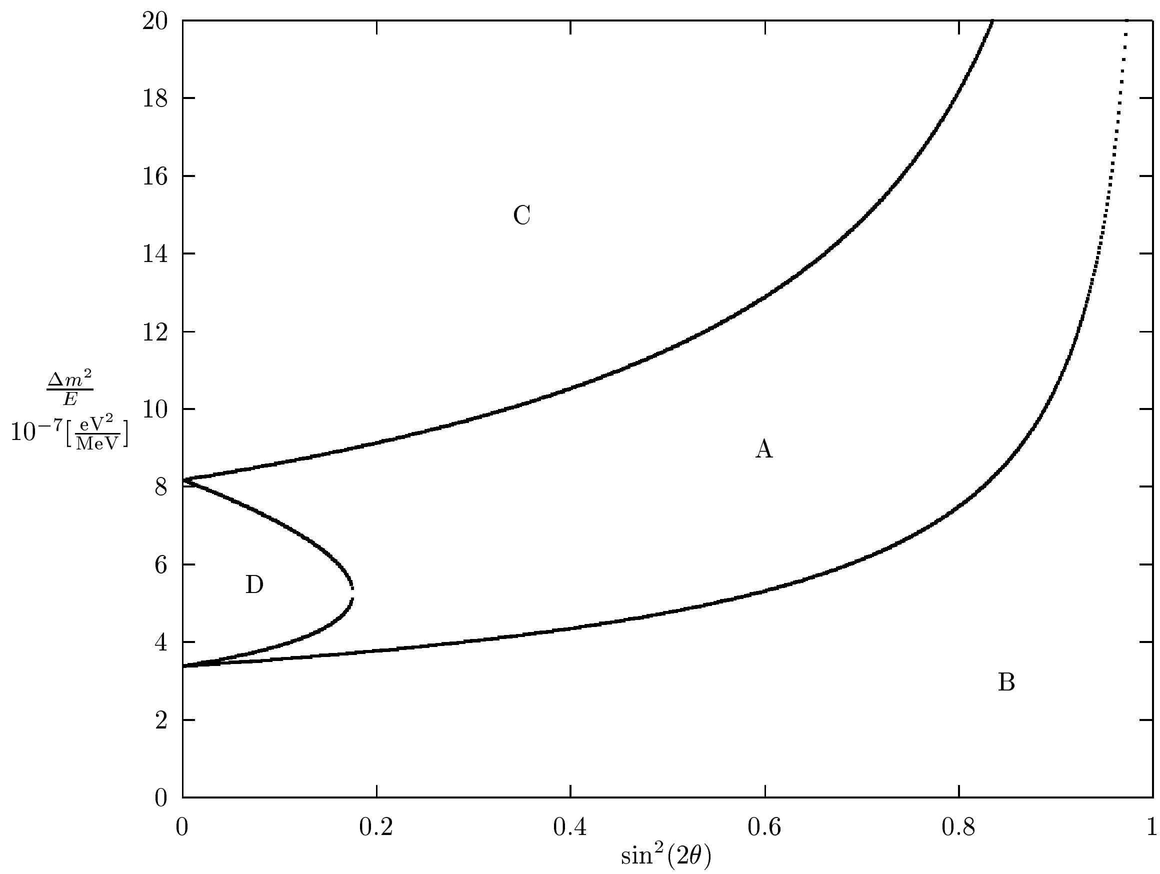

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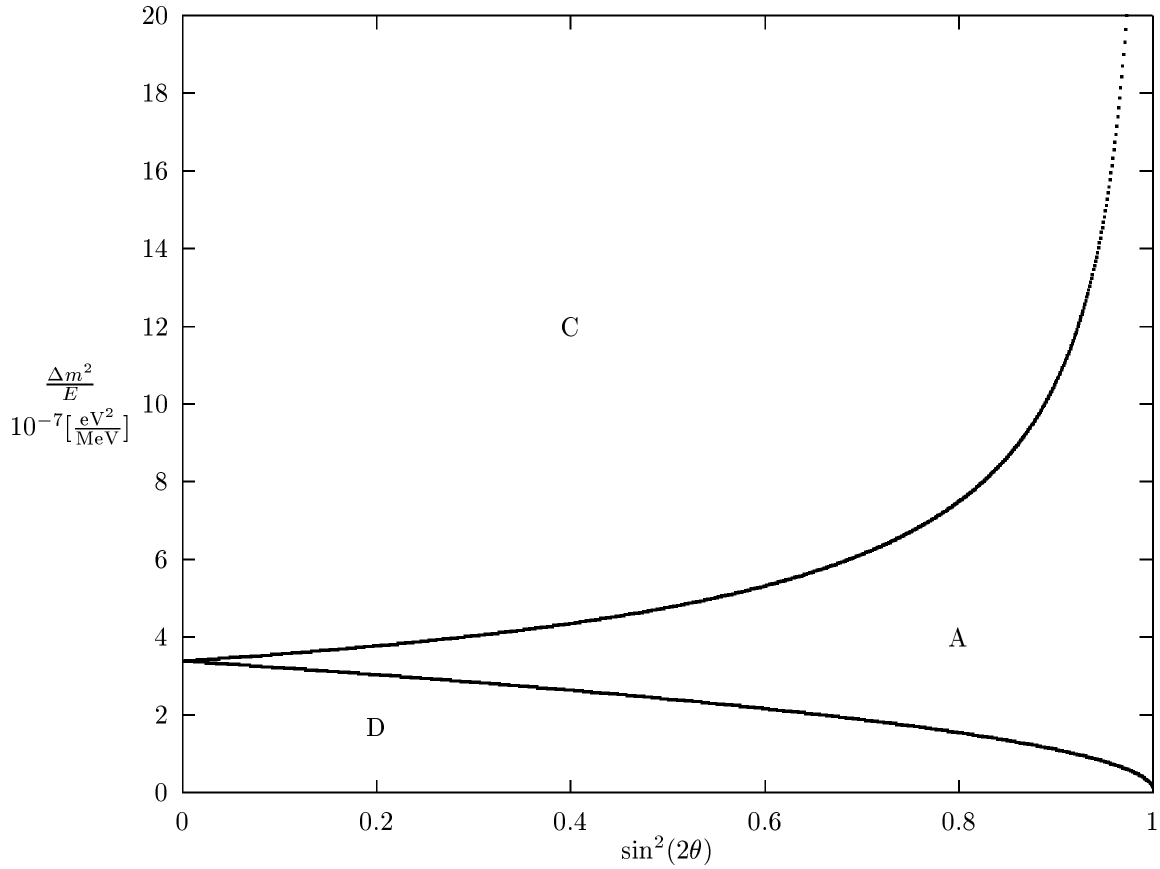

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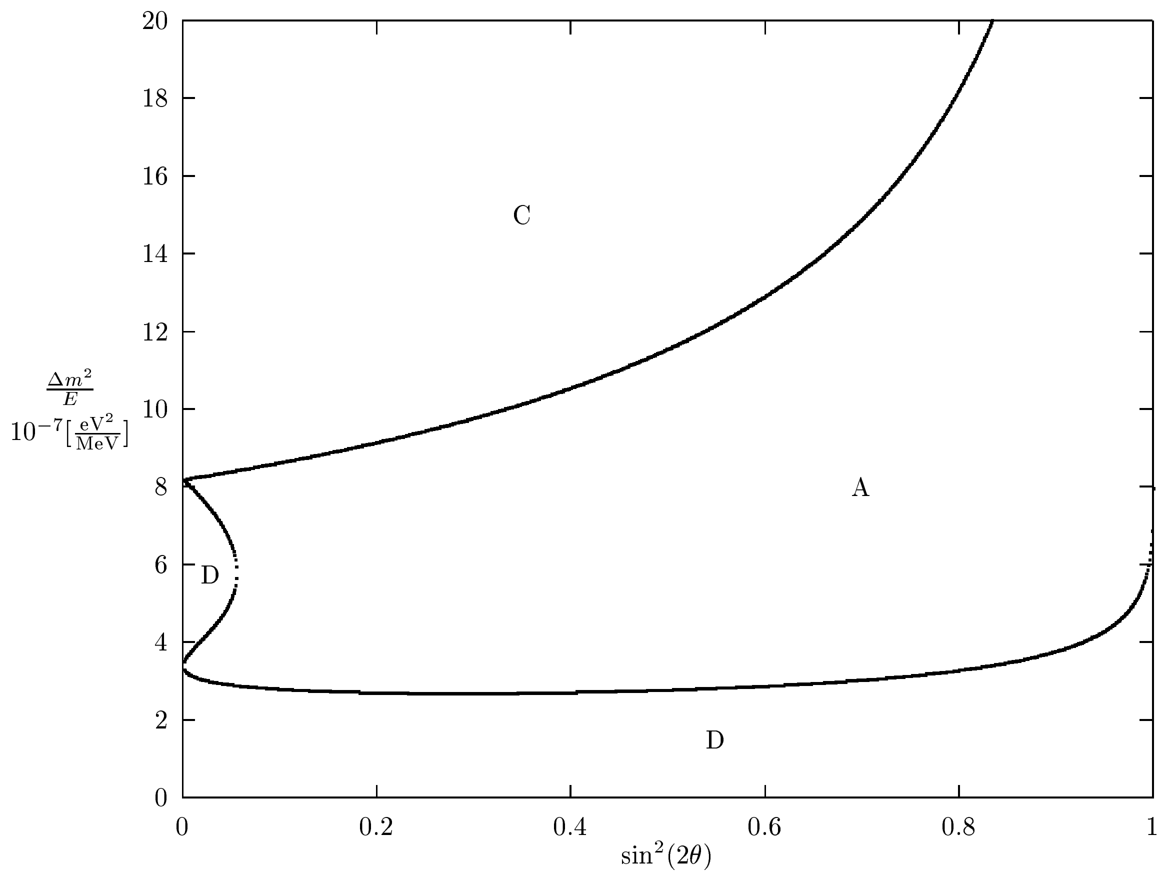

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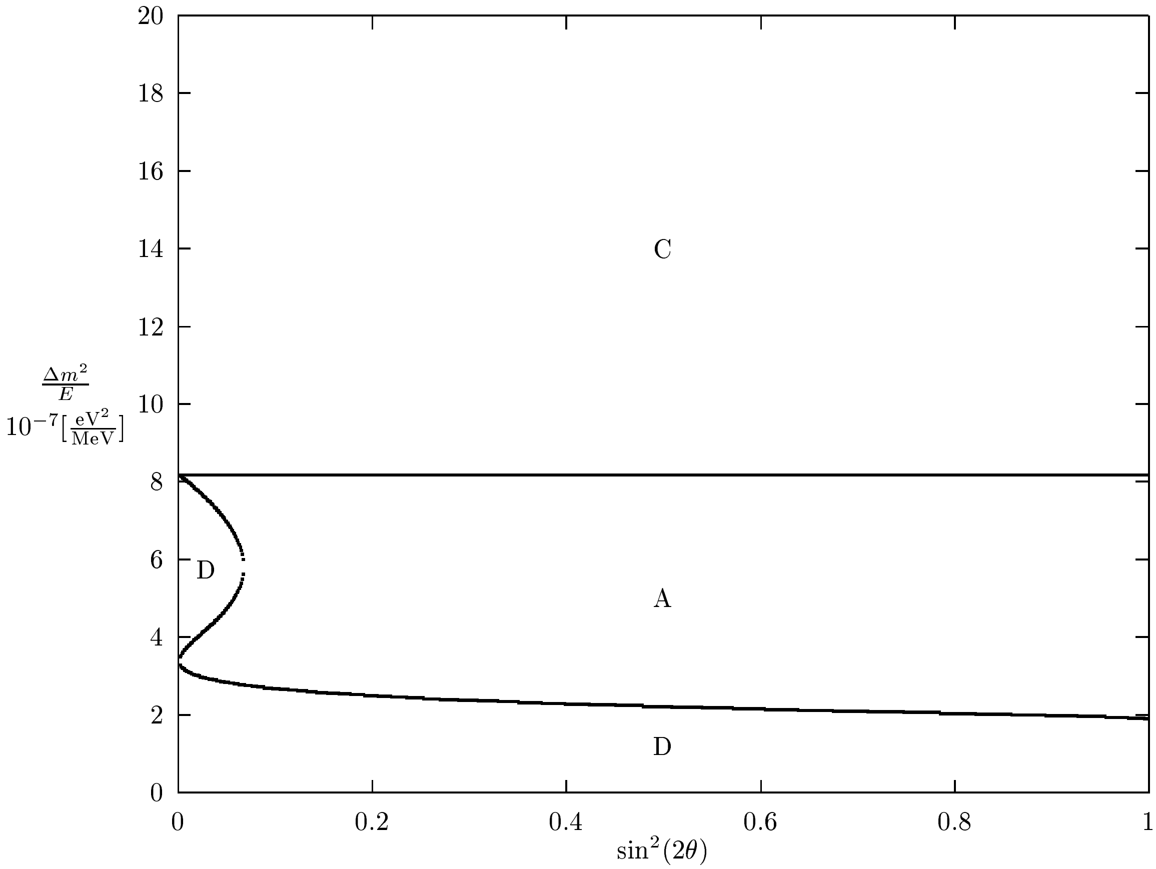

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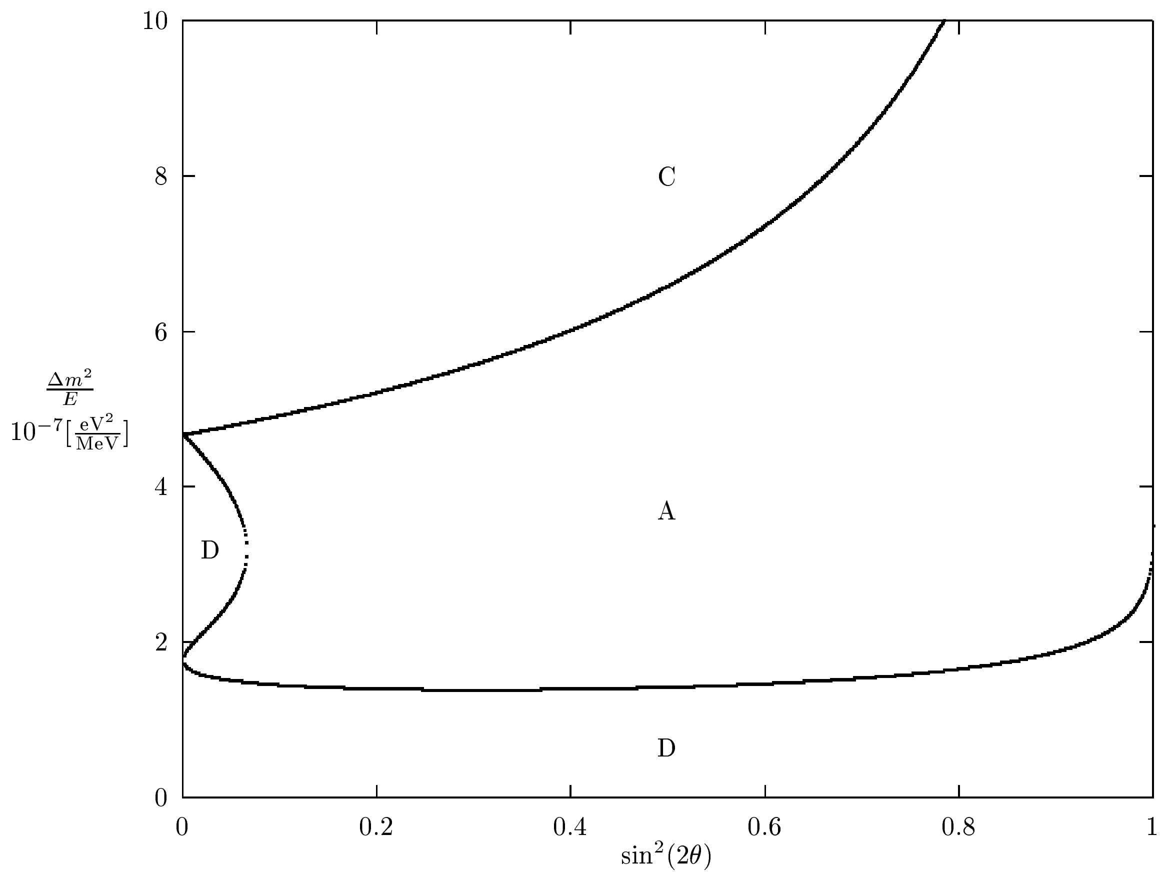

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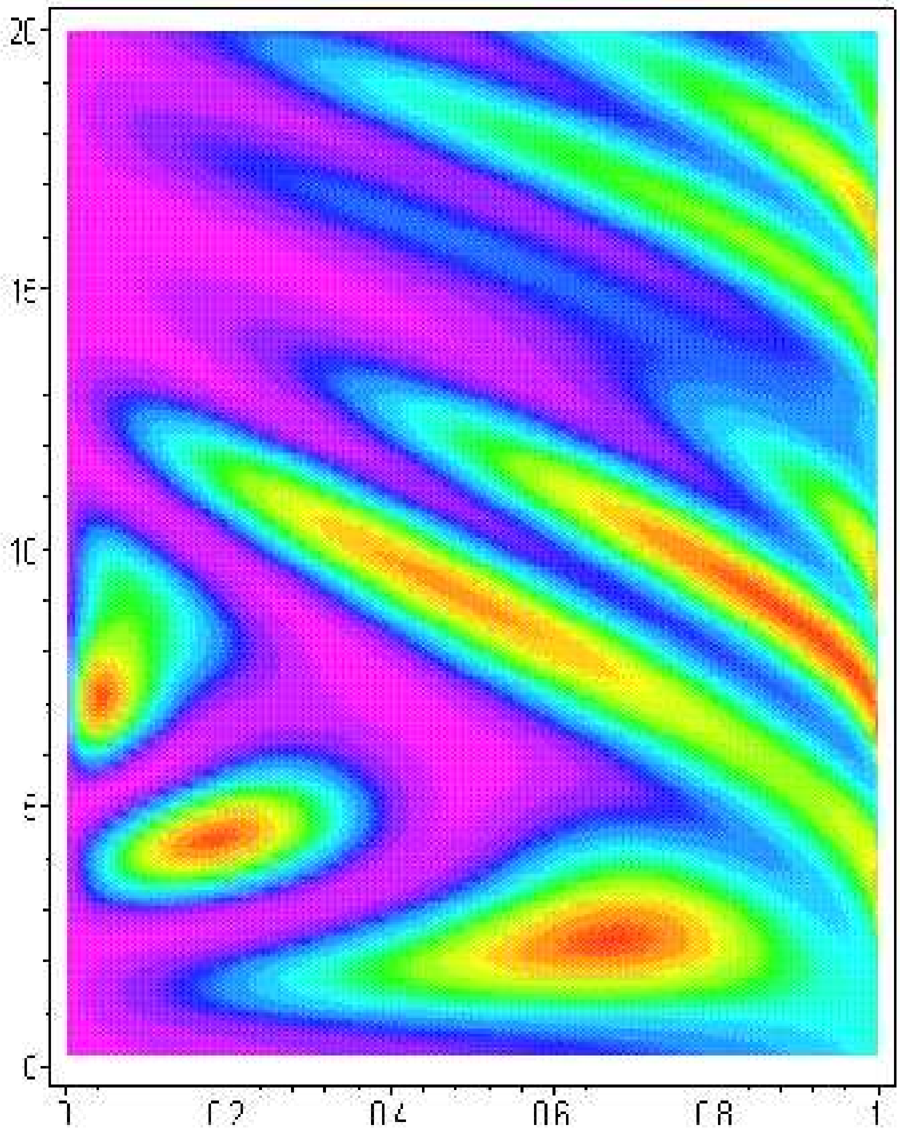
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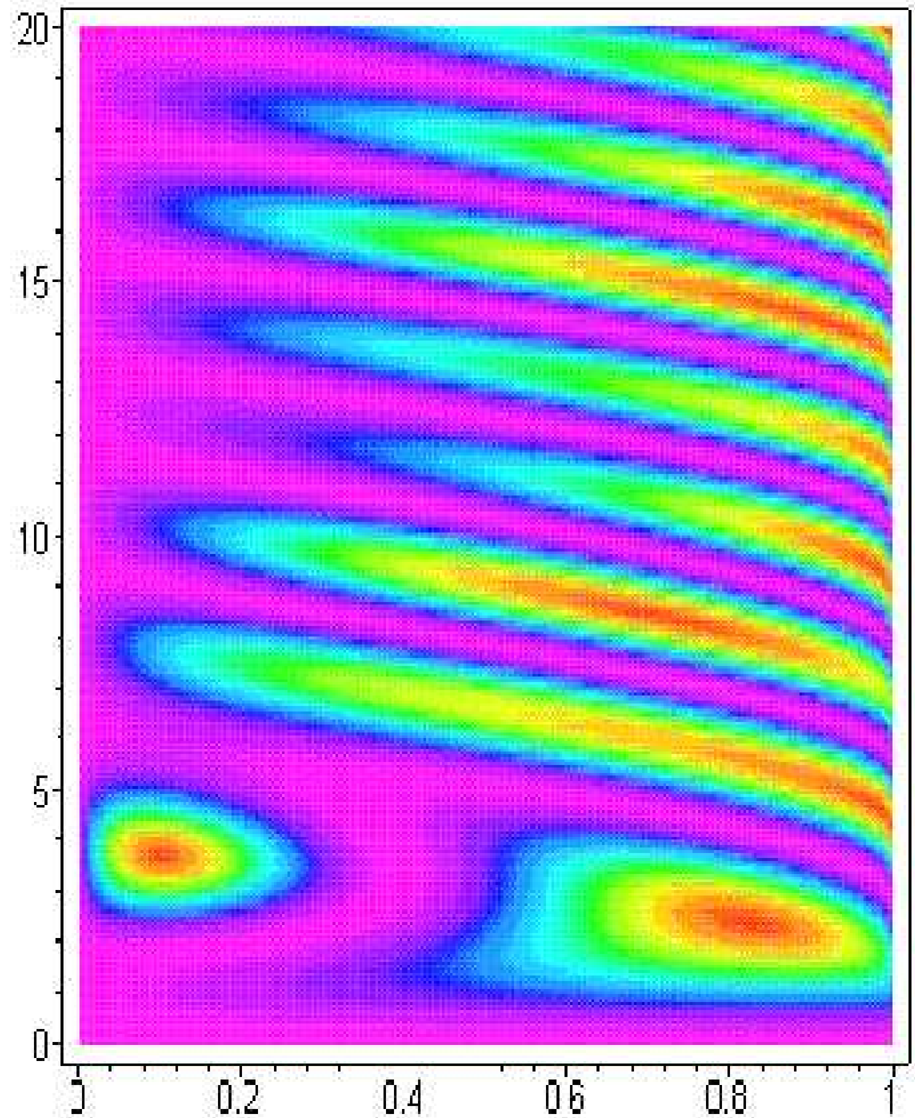
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